Math 1A Quiz 3 Solutions

October 12th, 2007

1. Consider the equation
$$y^{2x} = e^{x+\cos x}.$$
Use logarithmic differentiation to find y' in terms of x and y. You may assume $y > 0$.

Solution: Take ln of both sides:
$$\ln(y^{2x}) = \ln(e^{x+\cos x}).$$
Apply rules for ln to get:
$$2x \ln y = x + \cos x.$$
Now take derivatives with respect to x:
$$2\ln y + 2x\frac{1}{y}y' = 1 - \sin x.$$
Simplify to get:
$$y' = \frac{y - y\sin x - 2y\ln y}{2x}.$$

2. For each of the following functions f, state whether f has an inverse. If no, explain why not. If yes, find $f^{-1}(2)$.

Solutions:

a) $f(x) = x^2 + 2x + 1$**:** Not invertible. Factor to get
$$f(x) = (x+1)^2.$$
This is just a the parabola $y = x^2$ shifted one unit to the left. This fails the horizontal line test, so is not invertible.

b) $f(x) = 2x + \sin(x-1)$**:** Invertible. To see this, note that
$$f'(x) = 2 + \cos(x-1) \geq 1 \text{ for all } x.$$
Hence f is strictly increasing and continuous, so it must be one-to-one. We find $f^{-1}(2)$ by guess-and-check: $f(1) = 2 \cdot 1 + \sin(1-1) = 2 + 0 = 2$, so $f^{-1}(2) = 1$.

c) $f(x) = e^{(e^x)}$**:** Invertible. If $g(x) = e^x$, then $f = g \circ g$. The function g is invertible, so f is the composition of two invertible functions, hence invertible. We find $f^{-1}(2)$ by working backwards: set

$$2 = e^{(e^x)}$$

and take ln twice to get:

$$\ln(\ln 2) = x,$$

so $f^{-1}(x) = \ln(\ln 2)$.

3. Calculate

$$\lim_{x \to \frac{\pi}{2}} \frac{\ln|\sin x|}{x - \frac{\pi}{2}}.$$

Solution: Let $f(x) = \ln|\sin x|$. Then $f(\pi/2) = \ln|\sin \pi/2| = \ln 1 = 0$, so we can rewrite this limit as:

$$\lim_{x \to \frac{\pi}{2}} \frac{f(x) - f(\pi/2)}{x - \pi/2},$$

which is the definition of $f'(\pi/2)$. Using the fact that $\frac{d}{dx} \ln|x| = \frac{1}{x}$ and the chain rule, we see

$$f'(x) = \frac{1}{\sin x} \cos x = \cot x,$$

so $f'(\pi/2) = \cot(\pi/2) = 0$. So our answer is 0.

Bonus Point: Find two functions f and g, defined for all real numbers, such that $f \circ g$ is invertible, but neither f nor g is invertible.

Solution: There is no solution! Sorry! Let's see why: suppose g is not one-to-one. Then there are some real numbers x_1 and x_2 with $x_1 \neq x_2$ but $g(x_1) = g(x_2)$. Then

$$(f \circ g)(x_1) = f(g(x_1)) = f(g(x_2)) = (f \circ g)(x_2),$$

so $f \circ g$ is also not one-to-one. The question I *meant* to ask was to find two non-invertible functions f and g with $f \cdot g$ invertible. We'll see an answer to this on Monday. As far as the bonus point, there will be an extra opportunity for a bonus point at some later date to make up for this.