Math 1A Quiz 2

September 7th, 2007

Name ____________________ SID ________________

1. Answer the following questions True or False. Use T for true and 0 for false.

   a) $\mathbf{e^{xy} = e^x + e^y}$ **for all** x **and** y**.** False. For example, $e^{0\cdot 1} = e^0 = 1 \neq 1 + e = e^0 + e^1$.

   b) $\sin\left(\mathbf{x} + \frac{\pi}{\mathbf{2}}\right) = \cos \mathbf{x}$ **for all** x**.** True. Look at the graphs, for example.

   c) Every polynomial has a real zero. False. For example, $x^2 + 1$ has no real zeros.

   d) There is at least one real number a **such that** $\mathbf{e^a = \sin a}$**.** True. In fact, there are infinitely many; this is apparent from looking at the graphs of e^x and $\sin x$, or by applying the intermediate value theorem to $e^x - \sin x$.

   e) If f **and** g **are two functions and** $\lim_{\mathbf{x}\to\mathbf{a}} (\mathbf{f(x) + g(x)})$ **exists, then** $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f(x)}$ **exists and** $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g(x)}$ **exists.** False. As a counterexample, take any function $h(x)$ which does not have a limit at a, and let $f(x) = h(x)$ and $g(x) = -h(x)$.

   f) If f **is a function and** $|f|$ **is continuous, then** f **is continuous.** False. This was on the homework and discussed in class;

   $$f(x) = \begin{cases} -1 & , \quad x \leq 0 \\ 1 & , \quad x > 0 \end{cases}$$

   is a counterexample.

   g) $\lim_{\mathbf{x}\to\mathbf{0}} \mathbf{x} \sin\left(\frac{\mathbf{1}}{\mathbf{x}}\right) = \mathbf{0}$. True, for example by the squeeze theorem.

   h) If a function is continuous everywhere, then it is differentiable everywhere. False, $f(x) = |x|$ is a counterexample.

   i) If a function f **is continuous everywhere, then** $\lim_{x\to a} f(x)$ **exists for every real number** a**.** True. This is part of the definition of continuity.

   j) If a function is differentiable everywhere, then it is continuous everywhere. True. Proved in book, or note that if f is not continuous at a, then the denominator of

   $$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

goes to 0, while the numerator does not, so the limit does not exist, and hence f is not differentiable at a.

2. Using the limit laws and our rules for trigonometric functions, calculate

$$\lim_{x \to 0} \frac{\sin 2x}{2 \sin x}.$$

Solution:

$$\frac{\sin 2x}{2 \sin x} = \frac{\frac{sin2x}{2x}}{\frac{\sin x}{x}} \to \frac{1}{1} = 1.$$

Alternative method:

$$\frac{\sin 2x}{2 \sin x} = \frac{2 \sin x \cos x}{2 \sin x} = \cos x \to 1.$$

3. Let $f(x) = \frac{1}{x}$. Using the limit definition of the derivative, find $f'(x)$. What is the slope of the line tangent to the graph of $f(x)$ at $(1, 1)$?

Solution:

$$\frac{\frac{1}{x} - \frac{1}{a}}{x - a} = \frac{\frac{a-x}{ax}}{x - a} = \frac{-1}{ax} \to \frac{-1}{a^2},$$

so $f'(x) = -x^{-2}$. The $a + h$ definition works equally well. The slope of the tangent at $(1, 1)$ is just $f'(1) = -1$.

4. Bonus Point: Find a function f whose domain is all real numbers, such that f is not continuous anywhere, but f^2 is differentiable everywhere.

Solution:

$$f(x) = \begin{cases} 1 & , \quad x \text{ irrational} \\ -1 & , \quad x \text{ rational} \end{cases}$$

works.