Math 1A Quiz 1 Key

September 7th, 2007

1. Graph the function $f(x) = \cos(3x+1)$, and show carefully (for example, using a sequence of graphs) how to obtain this graph from the graph for $\cos(x)$.

   Answer: In words, we take the graph of $\cos(x)$, shift it one unit to the left, and then compress horizontally by a factor of 3.

   (Equivalently, compress horizontally by a factor of 3, then shift $\frac{1}{3}$ to the left.)

   Explanation: The changes we make are standard, the hard part is seeing which order we do this in.

   One way to see this: write $g(x) = \cos(x+1)$. We know the graph of g is the graph of $\cos(x)$ shifted one unit to the left. Now, $g(3x) = \cos(3x+1) = f(x)$, so the graph of f is the graph of g compressed by a factor of 3. This shows that we get the graph of f *first* by shifting, *then* by compressing.

   Now, if you weren't sure this was right, you should plug in values to check! If we take the graph of $\cos(x)$, shift one unit to the left, then compress by a factor of 3, we get a graph that has a zero at $\frac{\pi}{6} - \frac{1}{3}$. We can plug this into f to make sure f really has a zero here. We get:

$$\begin{aligned} f\left(\frac{\pi}{6} - \frac{1}{3}\right) &= \cos\left(3 \cdot \left(\frac{\pi}{6} - \frac{1}{3}\right) + 1\right) \\ &= \cos\left(\frac{\pi}{2} - 1 + 1\right) \\ &= \cos\left(\frac{\pi}{2}\right) = 0. \end{aligned}$$

   This tells us we probably did the right thing. A lot of people made the changes in the backwards order. Plugging in a value to check would have caught this error.

2. Let

$$g(x) = \frac{x^{100}}{x^2 + 5x + 6}$$

and

$$h(x) = e^x.$$

Find the domain of $h \circ g$.

[Hint: first find the domain of g.]

Answer: Factoring the denominator of g into $(x+2)(x+3)$, we see that g is defined for $x \neq -2, -3$, that is, for x in $(-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$. The function $h(x) = e^x$ is defined for all real numbers. Hence

$$(h \circ g)(x) = e^{\frac{x^{100}}{x^2+5x+6}}$$

is defined whenever g is defined, that is, for $x \neq -2, -3$.

Problems: Many people had trouble with the interval notation here: don't forget that $(-3, -2)$ interval in the middle! You need to know how to find the domain of a composition of two functions. Let's see how this works. Have g and h be any two functions. We have to remember that

$$(h \circ g)(x) = h(g(x)).$$

Now, for $h(g(x))$ to make any sense, $g(x)$ certainly has to be defined, so x must be in the domain of g. And for $h(g(x))$ to make sense, we need $g(x)$ to be in the domain of h.

In the case of $h(x) = e^x$, the domain of h is all real numbers, so the condition that $g(x)$ be in the domain of h is always true. Hence the domain of $h \circ g$ will just be the domain of g.

3. Find a $\delta > 0$ such that for $|x| < \delta$, we have

$$|e^x - 1| < \frac{1}{2}.$$

You may use the fact that $\ln 3 - \ln 2 < \ln 2$.

[Bonus point: prove this fact.]

Answer: We work backwards. First, we use our standard absolute value trick to see that $|e^x - 1| < \frac{1}{2}$ is the same as

$$-\frac{1}{2} < e^x - 1 < \frac{1}{2},$$

which is true if and only if

$$\frac{1}{2} < e^x < \frac{3}{2}.$$

Applying ln, we see that this will be true if and only if

$$\ln(1/2) < x < \ln(3/2).$$

And $\ln(1/2) = -\ln 2$ and $\ln(3/2) = \ln 3 - \ln 2$.
So, we want to pick a δ so that if $|x| < \delta$, we get

$$-\ln 2 < x < \ln 3 - \ln 2.$$

Since $|x| < \delta$ is the same as $-\delta < x < \delta$, we should take a δ so that $-\ln 2 \leq -\delta$ and $\delta \leq \ln 3 - \ln 2$ (this is our usual trick: we take the smaller distance from $a = 0$ to the endpoints of $(-\ln 2, \ln 3 - \ln 2)$).
In other words, we want $\delta \leq \min(\ln 2, \ln 3 - \ln 2)$. Since we were given $\ln 3 - \ln 2 < \ln 2$, we take $\delta = \ln 3 - \ln 2$.

Bonus Point: I saw several really cool ideas here. The easiest proof is probably:
Since $\frac{3}{2} < 2$ and ln is increasing,

$$\ln(3/2) < \ln 2.$$

And $\ln(3/2) = \ln 3 - \ln 2$, so

$$\ln 3 - \ln 2 < \ln 2.$$

Note: In our usual language, we have $f(x) = e^x$, $a = 0$, $L = 1$, and $\epsilon = \frac{1}{2}$.