Math 1A Practice Final Solutions

December 14th, 2007

1. Answer the following questions True or False. Use T for true and 0 for false. If false, give a counterexample.

a) Every continuous function is differentiable.

False. Counterexample: the function $f(x) = |x|$ is continuous but is not differentiable at 0. The converse of the statement *is* true: every differentiable function is continuous. Think of it this way: the graph of a continuous function has no "breaks." The graph of a differentiable function is "smooth." Everything which is smooth has no breaks, but not everything which has no breaks is smooth.

b) The product of two everywhere-differentiable strictly increasing functions is strictly increasing.

False. Counterexample: $f(x) = g(x) = x$ is a fine counterexample; both f and g are strictly increasing, but fg is decreasing for $x < 0$. There are examples where fg is strictly decreasing for all x; for example, $f(x) = e^{2x}$, $g(x) = -e^{-x}$.

c) Every continuous function has an antiderivative.

True. This is part of the fundamental theorem of calculus. If $f(x)$ is a continous function, then $F(x) = \int_{0^x} f(t)\, dt$ is an antiderivative for f.

d) Every integrable function is continuous.

False. Any piecewise-continous function which is not continuous is a counterexample (for example, the Heaviside function, or the greatest integer function). Integrable means "has an area under it." Continuous means "has no breaks." As a function, your stairs have breaks in them, but they still have area under them.

e) If f and g are two functions with the same antiderivative, then $f = g$.

True. If F is an antiderivative for both f and g, then $f = F' = g$. If you got this wrong, it's because you were thinking of the theorem that says that if f and g are defined on the same interval and $f' = g'$ then f and g differ by a constant.

2. Show that the function $f(x) = e^x - x - 2$ has exactly two roots.

Solution: We show that f has at most two roots and that f has at least two roots.

To show at *most* two roots, we use the Mean Value Theorem (in this case, Rolle's Theorem is equivalent). Note that $f'(x) = e^x - 1$ has exactly one zero at $x = 0$. Suppose f had three zeros, $a < b < c$. Then by the MVT, f' would have a zero between a and b and another between b and c. Hence f' would have at least two zeros. It doesn't, so f can't have more than 2 zeros.

To show at *least* two roots, we use the Intermediate Value Theorem. Note that $f(-2) = e^{-2} > 0$, $f(0) = -1 < 0$ and $f(2) = e^2 - 4 > 0$. Hence the IVT says f must have a root between -2 and 0 and another root between 0 and 2. Thus f has at least two roots.

3. Evaluate the following limits:
a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{8}{n} \sin\left(1 + \frac{2i}{n}\right)$.

Solution: This is a Riemann sum for some function; the simplest one comes from taking $\frac{b-a}{n} = \Delta x = \frac{2}{n}$, which we guess by looking at $\frac{2i}{n}$ inside of sin. Hence we guess $b - a = 2$, and from the 1 inside sin, we guess $a = 1$. We want to see a Δx multiplied with sin. We don't quite: we see instead $\frac{8}{n} = 4\Delta x$. Pull the 4 to the outside and now we have an honest Riemann sum for

$$4 \int_1^3 \sin(x)\, dx,$$

which is of course $4(-\cos(3) + \cos(1))$.

b) $\lim_{n\to\infty} \frac{\sqrt{3n^4+n^3}+\sin n}{n^2+2n-1}$.

Solution: Divide top and bottom by n^2. We have $\frac{1}{n^2} = \sqrt{\frac{1}{n^4}}$, so our limit becomes:

$$\lim_{n\to\infty} \frac{\sqrt{3n^4 + n^3} + \sin n}{n^2 + 2n - 1} \cdot \left(\frac{1/n^2}{1/n^2}\right) = \lim_{n\to\infty} \frac{\sqrt{3 + \frac{3}{n}} + \frac{\sin n}{n^2}}{1 + \frac{2}{n} - \frac{1}{n^2}}.$$

But now every term goes to 0 besides the $\sqrt{3}$ on the top and the 1 on the bottom, and we get that the limit is equal to $\sqrt{3}$.

c) $\lim_{x\to 0} \frac{\arccos(x)-1}{\arcsin(x)}$.

Solution: By continuity of $\arccos(x)$ and $\arcsin(x)$ at 0, we have that $\arccos(x) \to \frac{\pi}{2}$ and $\arcsin(x) \to 0$ as $x \to 0$. As we approach 0 from below, arcsin approaches 0 from below, and the limit is $-\infty$. As we approach from above, arcsin approaches 0 from above, and the limit is ∞. Hence the limit does not exist.

4. Simplify/evaluate the following expressions:
a) $\int \frac{1+x}{\sqrt{1+x^2}}\, dx$.

Solution: Break this into two integrals:

$$\int \frac{1}{\sqrt{1+x^2}}\, dx + \int \frac{x}{\sqrt{1+x^2}}\, dx.$$

The first integral is $\sinh^{-1}(x)$, by memorization. For the second, the u-substitution $u = 1+x^2$ gives the integral $\sqrt{1+x^2}$. Hence the whole expression is equal to

$$\sinh^{-1}(x) + \sqrt{1+x^2} + C.$$

b) $\frac{d^2}{dx^2} \int_0^{x^2} \left(\int_0^t \sqrt{1+u^5}\, du \right) dt$. [Note: here "simplify" means "get rid of the derivative." Your answer will still have an integral in it.]

Solution: To simplify our lives (and our notation), write $f(t) = \int_0^t \sqrt{1+u^5}\, du$. By the fundamental theorem of calculus, f is a differentiable function wherever it is defined. In particular, f is continuous for $t \geq 0$. Our expression is equal to:

$$\frac{d}{dx}\left(\frac{d}{dx} \int_0^{x^2} f(t)\, dt \right).$$

As we have done several times now, we apply both the fundamental theorem of calculus and the chain rule to the inside expression to get that this is equal to

$$\frac{d}{dx}\left(2xf(x^2)\right) = \frac{d}{dx}\left(2x \int_0^{x^2} \sqrt{1+u^5}\, du \right).$$

Now we apply the product rule, the fundamental theorem of calculus, and the chain rule to get that this is

$$2\int_0^{x^2} \sqrt{1+u^5}\, du + 2x\left(2x\sqrt{1+x^{10}}\right).$$

c) $\int_0^{\sqrt{3}} \sqrt{4 - x^2}\, dx$.

Solution: Interpret this as an area; it's the area between 0 and $\sqrt{3}$ under the semicircle of radius 2 centered at the origin. Plugging in $\sqrt{3}$ into $\sqrt{4-x^2}$, we see that the height of this semicircle at $\sqrt{3}$ is 1. Draw the line connecting the origin with $(\sqrt{3}, 1)$; this line cuts the area we are interested in into two regions: a sector of a circle, and a triangle. The area of the triangle is just $\frac{\sqrt{3}}{2}$ (using $\frac{1}{2}bh$). Noting that the triangle is a 30-60-90-triangle (with small angle adjacent to the origin) we see that the angle of the sector of the circle is $\frac{\pi}{3}$, i.e. this sector is one sixth of the whole circle. Hence it has area $\frac{4\pi}{6}$. Thus the integral is

$$\frac{\sqrt{3}}{2} + \frac{2\pi}{3}.$$

d) $\int_0^{\pi/2} (\sin^3 x \cos x) e^{\cos^2 x}\, dx$. [Hint: $ue^u - e^u$ is an antiderivative for ue^u.]

Solution: Use the substitution $u = \cos^2 x$, $du = -2\cos x \sin x dx$. Then $\sin^2 x = 1 - \cos^2 x = 1 - u$, and we have that

$$\begin{aligned}
\int_0^{\pi/2} (\sin^3 x \cos x) e^{\cos^2 x}\, dx &= \frac{-1}{2} \int_0^{\pi/2} (-2 \sin x \cos x)(\sin^2 x) e^{\cos^2 x}\, dx \\
&= \frac{-1}{2} \int_1^0 (1-u) e^u\, du \\
&= \frac{-1}{2} \left[e^u - ue^u + e^u \right]_1^0 \\
&= \frac{e}{2} - 1
\end{aligned}$$

Note the change in endpoints because of substitution.

5. Use the "shell method" to find the volume of the solid generated by rotating the region bounded by $y = \ln x$, $y = 0$, $y = 1$, and $x = 0$ around the x-axis. [Hint: $te^t - e^t$ is an antiderivative for te^t].

Solution: Set this up like any normal shell method problem. You'll know you did it right if you get volume given by

$$2\pi \int_0^1 ye^y\, dy = 2\pi \left[ye^y - e^y \right]_0^1 = 2\pi.$$