A "CHEAT SHEET" TO PROOFS ABOUT SETS

1. The Basics

Implications: To show "if A then B," there are two basic methods:

- **Direct Proof:** Assume A is true, then show B is true.
- **The Contrapositive:** Assume B is false, then show A is false.

Equivalences: To show "A if and only if B," prove "A implies B" and "B implies A."

Conjunctions: To show "A and B," prove A, then prove B.

Disjunctions: To show "A or B," assume A is false, then show B is true. Or go the other way around.

Proving the Negative: To show A is **false**, there are two basic methods:

- **Giving a Counterexample:** If A is a statement of the form "all B's are C's," give an example of a B which is not a C.
- **Proof by Contradiction:** Assume A is true, and use this to prove a statement that you know is false.

Universal Quantification: To prove a statement of the form "all A's are B's," let a be an arbitrary A, then show that a is a B.

Existential Quantification: To prove a statement of the form "there is an A that is a B," give an example of an A which is a B.

2. The Negations of the Basics

Negation of Conjunction: Proving "not (A or B)" is the same as proving "(not A) and (not B)" (note the "or" changed to an "and").

Negation of Disjunction: Proving "not (A and B)" is the same as proving "(not A) or (not B)."

Negation of Implication: Proving "not (A implies B)" is the same as proving "A and (not B)."

Negation of Universal Quantification: Proving "not (every A is a B)" is the same as proving "there is an A that is not a B."

Negation of Existential Quantification: Proving "not (there is an A that is a B)" is the same as proving "every A is not a B."

An example: : suppose you want to disprove the statement "every integer is the square of an integer." This can be restated as "for every n, if n is an integer then n is the square of an integer." Now, we want to prove "not (for every n, if n is an integer then n is the square of an integer." By the rule for negating universal quantification, this is the same as proving that "there is an n such that not (if n is an integer then n is the square of an integer." By the rule for negating implication, this is the same as proving that "there is an n such that n is an integer and n is not a square of an integer." So we just need to give an example of such an integer. The integer 2 is such an example (on HW1 you'll prove that in fact 2 is not even the square of a rational number).

3. Proofs about Sets

Let X and Y be sets.

Subsets: To show $X \subseteq Y$ is the same as to show "all elements of X are elements of Y." So by "Universal Quantification" above, to show $X \subseteq Y$, let $x \in X$ be arbitrary, and show $x \in Y$.

Set Equality: To show $X = Y$, prove $X \subseteq Y$ and $Y \subseteq X$.

Set-Builder Notation: There are two kinds:

- **Subset defined by a rule:** Suppose P is a property of elements of X. To show $a \in \{x \in X \mid P(x) \text{ is true}\}$, show that 1) $a \in X$ and 2) $P(a)$ is true.
- **Outputs of a function:** Suppose $f : X \to Y$ is a function. To show $a \in \{f(x) \mid x \in X\}$, show that there is an $x \in X$ such that $f(x) = a$.

4. Proofs about Functions

Let X and Y be sets and let $f : X \to Y$ be a function.

Function Equality: Let $g : X \to Y$ be another function. To show $f = g$, we need to show $f(x) = g(x)$ for all $x \in X$. By "Universal Quantification," we prove this by letting $x \in X$ be arbitrary and showing $f(x) = g(x)$.

Inverses: Let $g : Y \to X$ be another function. To show $g = f^{-1}$, we need to show $(g \circ f) = 1_X$ and $(f \circ g) = 1_Y$. By "Function Equality," we prove this by showing that for all $x \in X$, $(g \circ f)(x) = x$, and for all $y \in Y$, $(f \circ g)(y) = y$.

Injectivity: To show that f is injective, we need to show that for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. So, we let $x_1, x_2 \in X$ be arbitrary, and we assume that $f(x_1) = f(x_2)$. Then we prove that $x_1 = x_2$.

Surjectivity: To show that f is surjective, we need to show that for all $y \in Y$, there is an $x \in X$ such that $f(x) = y$. So, we let $y \in Y$ be arbitrary, then we find an $x \in X$ such that $f(x) = y$.

Bijectivity: To show that f is bijective, we prove that f is injective and f is surjective. Alternatively, we can find an inverse to f.

5. Relations

Let X be a set and let R be a relation on X, i.e. let R be a subset of $X \times X$. Recall we write xRy to mean $(x, y) \in R$.

Reflexivity: To prove that R is reflexive, let $x \in X$ be arbitrary, then show that xRx.

Symmetry: To prove that R is symmetric, let $x, y \in X$ be arbitrary, assume that xRy, then show that yRx.

Transitivity: To prove that R is transitive, let $x, y, z \in X$ be arbitrary, assume that xRy and yRz, then show that xRz.

Equivalence Relations: To show that R is an equivalence relation, prove that R is reflexive, symmetric, and transitive.

Well-Definition: Assume that R is an equivalence relation, let Y be a set, and let $f : X \to Y$ be a function. To show that the function $g : X/R \to Y$ defined by $g([x]) = f(x)$ is well defined, let $x_1, x_2 \in X$ be arbitrary and assume that $x_1 R x_2$. Then show that $f(x_1) = f(x_2)$.