MATH 113 HOMEWORK 7
DUE MONDAY, AUGUST 10TH

1. Rings and Ideals

Problem 1.1. Let R be a ring and let I and J be ideals of R.

(1) Show that $I \cap J$ is an ideal of R.
(2) Show that the set $I + J = \{i + j \mid i \in I, j \in J\}$ is an ideal of R.
(3) Let $n, m \in \mathbb{N}$ be arbitrary, and consider the principal ideals $(n) \subseteq \mathbb{Z}$ and $(m) \subseteq \mathbb{Z}$. Calculate $(n) \cap (m)$ and $(n) + (m)$.
(4) Consider the principal ideals $I = (x)$ and $J = (x + 1)$ in $\mathbb{R}[x]$. Calculate $I \cap J$ and $I + J$.

Problem 1.2. Let $n \in \mathbb{N}$ be arbitrary. Show that the ideal $n\mathbb{Z} \subseteq \mathbb{Z}$ is prime if and only if n is prime.

2. Polynomials

Problem 2.1. Let I be the subset of $\mathbb{Z}[x]$ consisting of all polynomials with even constant term, i.e. if $p(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x]$ then $p(x) \in I$ iff $a_0 \in 2\mathbb{Z}$.

(1) Show that I is an ideal of $\mathbb{Z}[x]$.
(2) Show that I is not a principal ideal of $\mathbb{Z}[x]$.
(3) Is I prime? Maximal?

Problem 2.2. Let R be a commutative ring with 1 and let I be an ideal of R. Let $I[x] \subseteq R[x]$ be the set of all polynomials $a_0 + a_1x + \cdots + a_nx^n \in R[x]$ such that $a_i \in I$ for all i.

(1) Show that $I[x]$ is an ideal of $R[x]$.
(2) Show that $R[x]/I[x] \cong (R/I)[x]$.
(3) Use (2) to show that $3\mathbb{Z}[x]$ is a prime ideal in $\mathbb{Z}[x]$ which is not maximal.

3. Irreducible Polynomials and Fields

Problem 3.1. Which of the following polynomials are irreducible over $\mathbb{Q}[x]$?

(1) $x^3 + 3x^2 + x + 1$,
(2) $x^4 + x^2 + 2x + 1$,
(3) $5x^5 + 4x^4 + 6x^2 + 10x + 6$.

Problem 3.2. Let $p(x) = x^2 + ax + b \in \mathbb{R}[x]$ be an irreducible polynomial. Show that $\mathbb{R}[x]/(p(x)) \cong \mathbb{C}$. (Hint: quadratic formula!).

Problem 3.3 (An example of a finite field)**.** Do the following:

(1) Show that the polynomial x^2+1 is irreducible in $\mathbb{Z}/3\mathbb{Z}[x]$. Conclude that the ring $K = (\mathbb{Z}/3\mathbb{Z}[x])/(x^2+1)$ is a field.
(2) Show that every element of this field can be uniquely written in the form $\overline{ax+b}$ for some $a, b \in \mathbb{Z}/3\mathbb{Z}$, and use this to calculate the number of elements in K.
(3) Compute a full multiplication table for K.
(4) Let M be the 2×2-matrix

$$M = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

with entries in $\mathbb{Z}/3\mathbb{Z}$. Show that $M^2 = -I$, where I is the identity matrix.
(5) Consider the set of all matrices of the form $aM + bI$ where $a, b \in \mathbb{Z}/3\mathbb{Z}$. Show that this set forms a ring R.
(6) Show that $K \cong R$. (Hint: use the first isomorphism theorem).

Problem 3.4. Show that there exists a field with 8 elements. Compute a full multiplication table for your example.

4. Linear Algebra

Problem 4.1. Do Judson, Ch. 18, Problem 15.