MATH 113 HOMEWORK 6
DUE MONDAY, AUGUST 3RD

1. Abelian Groups

Problem 1.1 (Linear Algebra of Abelian Groups, Part I)**.** Let A be an abelian group, and suppose that $\{a_1, \ldots, a_n\} \in A$ is a generating set for A.

(1) Show that for $i \neq j$ and $k \in \mathbb{Z}$, the set $\{a_1, \ldots, a_{i-1}, a_i + ka_j, a_{i+1}, \ldots, a_n\}$ is also a generating set for A. Show moreover that this set is a basis for A if and only if $\{a_1, \ldots, a_n\}$ is.
(2) Show that for all i, the set $\{a_1, \ldots, a_{i-1}, (-a_i), a_{i+1}, \ldots, a_n\}$ is a generating set for A. Show moreover that this set is a basis if and only if $\{a_1, \ldots, a_n\}$ is.

Problem 1.2 (Linear Algebra of Abelian Groups, Part II)**. You are NOT required to write up this problem to submit. However, you must understand it completely, as you will need it for the next problem, and for the final exam, hint hint.**
Let H be a subgroup of $\mathbb{Z}^n$. We know that H is finitely generated; suppose $h_1, \ldots, h_k$ generate H. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{Z}^n$. We know that we can write each h_i as

$$h_i = (a_{1i}, a_{2i}, \ldots, a_{ni}) = \sum_{j=1}^{n} a_{ji} e_j.$$

Bundle this information as an $n \times k$-matrix $M = (a_{ij})$, so the ith column of M contains the coordinates of h_i.

(1) Show that exchanging the generators h_i and h_j corresponds to exchanging the ith and jth columns of M.
(2) Show that exchanging the basis vectors e_i and e_j corresponds to exchanging the ith and jth rows of M.
(3) Show that multiplying the generator h_i by -1 corresponds to multiplying the ith column of M by -1.
(4) Show that multiplying the basis vector e_i by -1 corresponds to multiplying the ith row of M by -1.
(5) For $i \neq j$ and $k \in \mathbb{Z}$, show that replacing the generator h_i by $h_i + kh_j$ corresponds to adding k times the jth column of M to the ith column of M.
(6) For $i \neq j$ and $k \in \mathbb{Z}$, show that replacing the basis vector e_i by $e_i + ke_j$ corresponds to *subtracting* k times the ith row of M from the jth row of M (note that the roles of i and j have also changed!).

Conclude that when calculating $\mathbb{Z}^n/H$, we may apply any of the above "elementary row and column operations" to M and obtain the same quotient.

Problem 1.3 (Linear Algebra of Abelian Groups, Part III)**.** Let's use the previous problem to actually do some calculations!

(1) Let H be the subgroup of $\mathbb{Z}^2$ generated by $(6, 9)$. Use the previous problem to show that $\mathbb{Z}^2/H \cong \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. By keeping track of the row operations you do, explain how to choose the basis of $\mathbb{Z}^2$ that gives this isomorphism.
(2) Let H be the subgroup of $\mathbb{Z}^3$ generated by $(1, 2, 3)$ and $(2, 2, 2)$. Calculate $\mathbb{Z}^3/H$ as a product of cyclic groups.
(3) Let H be the subgroup of $\mathbb{Z}^3$ generated by $\{(1, 2, 3), (3, 4, 5), (5, 6, 7), (7, 8, 9)\}$. Calculate $\mathbb{Z}^3/H$ as a product of cyclic groups.

Problem 1.4. Let A and B be finitely generated abelian groups. Show that $\text{rank}(A\times B) = \text{rank}(A)+\text{rank}(B)$. (Hint: use the structure theorem, plus the proposition proved on the last homework that $\text{rank}(A) = \text{rank}(A/T(A))$.)

2. Group Actions

Problem 2.1. Let $G = (\mathbb{R}, +)$, and define an action of G on $\mathbb{R}^2$ by letting $\theta \in G$ act by clockwise rotation by θ. Show that this is in fact an action of G. Find the orbits of this action (a geometric description will suffice), and for each $v \in \mathbb{R}^2$, compute the stabilizer of v.

Problem 2.2 (Creative Problem: Properties of Group Actions)**.** Let G be a group and let X be a G-set.

(1) The action of G on X is called *faithful* if the only element of G which acts trivially is the identity, i.e. if for every $g \in G$, $g \cdot x = x$ for all $x \in X$ only if $g = e$. Give three examples of faithful group actions and three examples of non-faithful group actions.
(2) The action of G on X is called *transitive* if it has a single orbit, i.e. if for every $x, y \in X$ there is a $g \in G$ such that $g \cdot x = y$. Give three examples of transitive group actions and three examples of non-transitive group actions.
(3) Suppose that the action of G on X is both faithful and transitive and suppose that X is non-empty. Show that there is a bijection from G to X.

Problem 2.3 (Permutation Representations)**.** Let G be a group and let X be a G-set. For each $g \in G$, define the function $\lambda_g : X \to X$ by $\lambda_g(x) = g \cdot x$. Do the following:

(1) Show that λ_g is a bijection, i.e. $\lambda_g \in S_X$.
(2) Show that the function $\lambda : G \to S_X$ defined by $\lambda(g) = \lambda_g$ is a homomorphism.
(3) Show that the function λ defined in part (2) is injective if and only if the action of G on X is faithful.

(4) Deduce Cayley's Theorem from parts (2) and (3) applied to the left regular representation of G.

Problem 2.4. How many different ways can the vertices of a hexagon be colored with the three colors red, green, and blue, up to *rotation* of the hexagon? What about if we allow all symmetries in D_6? Give an example of two colorings of the hexagon which are not equivalent in the first case, but are equivalent in the second case.

3. Rings

Problem 3.1 (The Group of Units)**.** Let R be a ring with unity. Show that the set of all elements in R which have a multiplicative inverse forms a group under multiplication, called the group of units of R. We denote this group by $R^\times$. What is $M_2^\times$? What is $\mathbb{Z}^\times$?

Problem 3.2. Do Judson, Ch. 14, Exercise 4.