MATH 113 HOMEWORK 5
DUE MONDAY, JULY 27TH

1. The Isomorphism Theorems

Problem 1.1. Let $C \leq \mathbb{C}^*$ be the circle group. Use the first isomorphism theorem to show that:

(1) $\mathbb{R}/\mathbb{Z} \cong C$.
(2) There is a subgroup of C which is isomorphic to $\mathbb{Q}/\mathbb{Z}$.
(3) $\mathbb{C}^*/C \cong (\mathbb{R}_{>0}, \cdot)$, the group of positive real numbers with multiplication.

Problem 1.2 (The Universal Property of the Quotient)**.** Let G be a group and let N be a normal subgroup of G. Let $\phi : G \to G/N$ be the canonical homomorphism.

(1) Let H be another group and let $f : G \to H$ be a homomorphism. Show that there is a homomorphism $\bar{f} : G/N \to H$ such that $\bar{f} \circ \phi = f$ if and only if $N \leq \ker(f)$. Moreover, show that $\bar{f}$ is unique if it exists.
(2) Use (1) to show that there is a surjective homomorphism $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ if and only if $m|n$.

Remark: Problem 1.2.(1) says that giving a homomorphism $g : G/N \to H$ is the same as giving a homomorphism $f : G \to H$ such that $N \subseteq \ker(f)$.

Problem 1.3. Let G be a group, let $N \lhd G$ be a normal subgroup, and let $K \leq G$ be another subgroup. Let $\phi : G \to G/N$ be the canonical homomorphism. Show that $\phi(K) = KN/N$ (i.e. show that $\phi(K)$ and KN/N are *the same subgroup* of G/N, not just isomorphic).

Problem 1.4. Do Judson, Ch. 9, Exercise 11.

Problem 1.5. Consider the group D_6, and let $H = \langle r^2 \rangle$ and $K = \langle r^3 \rangle$ in D_6. Show that H and K are both normal subgroups of D_6. Calculate D_6/H and D_6/K, and use this calculation to compute the subgroup lattice of D_6.

2. The Structure Theorem for Finitely Generated Abelian Groups

Problem 2.1. Do the following:

(1) List all abelian groups of order 24.
(2) List all abelian groups of order 81.
(3) Show that every abelian group of order 30 is cyclic.

Problem 2.2 (Invariant Factors)**.** Suppose that A is an abelian group of the form

$$A = \mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_n\mathbb{Z},$$

where $a_1|a_2|\cdots|a_n$. We know that A can be written uniquely as a product

$$A \cong \mathbb{Z}/p_1^{e_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{e_k}\mathbb{Z}.$$

Do the following:

(1) Explain how to recover the numbers $a_1, \ldots, a_n$ from $p_1^{e_1}, \ldots, p_k^{e_k}$.

(2) Elaborating on (1), explain how every finite abelian group can be written uniquely as

$$A \cong \mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_n\mathbb{Z}$$

for some $a_1, \ldots, a_n \in \mathbb{N}$ with $a_1|a_2|\cdots|a_n$.

(3) Carry out the process you described in (2) for the group

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}.$$

Problem 2.3. Let A be an abelian group. Show that $\text{rank}(A) = \text{rank}(A/T(A))$, where $T(A)$ is the torsion subgroup of A.