MATH 113 HOMEWORK 3
DUE MONDAY, JULY 13TH

1. Basic Computations

Proofs can be omitted in this section. You should still give enough explanation that your classmates could learn the technique from your calculations.

Problem 1.1. Do Judson Ch. 3 Exercises 20 and 24.

Problem 1.2. Consider the permutations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 4 & 3 & 1 & 5 & 2 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 5 & 7 & 6 & 2 \end{pmatrix}$$

in S_7. Do the following:

(1) Decompose σ and τ into cycles.
(2) Compute $\sigma\tau$ and $\tau\sigma$.
(3) Compute the order of σ, τ, $\sigma\tau$, and $\tau\sigma$.
(4) Determine the signs of σ, τ, $\sigma\tau$, and $\tau\sigma$.

Problem 1.3. Find all possible cycle structures in S_7. Explain which of these cycle structures correspond to elements of A_7. Use this to calculate all possible orders of elements of A_7.

Problem 1.4. Do Judson Ch. 5 Exercise 5.

2. Basic Problems

Problem 2.1. Let G be a group and let $a, b \in G$ be arbitrary. Show that:

(1) The order of a is equal to the order of $b^{-1}ab$.
(2) The order of ab is equal to the order of ba.

Problem 2.2. Let G be a group and let H and K be subgroups of G.

(1) Show that $H \cap K$ is a subgroup of G. Conclude that $H \cap K$ is also a subgroup of H and a subgroup of K.
(2) Suppose that H is cyclic of order 15 and K is cyclic of order 16. What is the order of $H \cap K$?

Problem 2.3. Let G be a group.

(1) Show that if G is abelian, then the set of elements of finite order in G form a subgroup of G. This is called the *torsion subgroup* of G.
(2) Give an example that shows that if G is not assumed to be abelian, then the set of elements of finite order in G can fail to be a subgroup of G.

Problem 2.4. (Judson Ch. 4 Exercise 6) Find all of the subgroups of A_4. Calculate the order of each subgroup you find. Notice that there is no subgroup of order 6. Explain why you might find this surprising.

Problem 2.5. (Judson Ch. 4 Exercise 18) Show that A_n is nonabelian for $n \geq 4$.

Problem 2.6. (Judson Ch. 4 Exercise 26) Show that any permutation in S_n can be written as a product of the transpositions $(12), (13), \ldots, (1n)$. Show likewise that it can be written as a product of the transpositions $(12), (23), \ldots, ((n-1)\, n)$. Finally, show that it can be written as a product of the two permutations (12) and $(12\ldots n)$.

3. Creative Problems

Problem 3.1. (Other groups of symmetries) We've now seen groups of "symmetries of a set" (the symmetric group S_n), "symmetries of the n-gon" (the dihedral group D_n), and a few other groups of symmetries (for example, the symmetries of a rectangle don't fall into either class above). Now consider the following:

(1) The "group of symmetries of the integers": Let G be the set of functions $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = ax + b$, where $a, b \in \mathbb{R}$, such that f induces a bijection from $\mathbb{Z}$ to $\mathbb{Z}$ (for example, $f(x) = x + 1$ is such a function, but $f(x) = 2x + 1$ is not). Describe all possible a and b. Show that G is a group under composition. Try to find as many interesting properties of G as you can.

(2) Now come up with TWO other interesting examples of "groups arising from symmetries"; you might look at symmetries of your favorite shape, or symmetries of the tile pattern on your bathroom wall. Try to say as much as you can about the groups you discover.

Problem 3.2. Let G be a group, let g be an element of G, and let H be a subgroup of G. Do the following:

(1) Show that $gH = Hg$ if and only if $gHg^{-1} = H$; here $gHg^{-1} = \{ghg^{-1} \mid h \in H\}$.

(2) Part (1) shows that multiplying subgroups of G by elements of G "behaves like multiplying elements of G" in some sense. Try to find other examples of this. Try to find cases where this fails (for example, what happens if you replace g here by another subgroup K of G?). See how general of a statement you can prove.