MATH 113 HOMEWORK 1
DUE MONDAY, JUNE 29TH

Directions: You are free to work in groups for all problems, but you should attempt each problem on your own first. You must submit your own set of solutions, which you should write individually. If you work in a group on a given problem, list the names of your group members above your solution to that problem.

Note: unless otherwise indicated, the phrase "give an example of an X with property Y" should always be read to mean "give an example of X with property Y and prove that your example really is an X and really has property Y."

1. Special Assignment

Problem 1.1. Come to my office hours this week – either Wednesday 2-3:30 or Thursday 2-3:30 (note the special time) – and introduce yourself. This problem will definitely be graded.

2. Basic Problems

Sets and Functions.

Problem 2.1. Let A, B, and C be sets. Prove that:

(1) $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$.
(2) $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$.
(3) Draw Venn diagrams that explain why these statements are true.

Problem 2.2. Let A, B, and C be sets and let $f : A \to C$ and $g : B \to C$ be functions. Define

$$D := \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

Let $h : D \to A$ be defined by $h(a, b) = a$ and let $k : D \to B$ be defined by $k(a, b) = b$.

(1) Show that $f \circ h = g \circ k$.
(2) Compute D explicitly in the case that $A = B = C = \mathbb{Z}$, $f(n) = 2n$, and $g(n) = n^2$.

Problem 2.3. Do Judson, Ch. 0, Exercises 17 and 24.

Relations.

Problem 2.4. Show that each of the following relations is an equivalence relation. In each case, identify the equivalence classes.

(1) The relation R on $\mathbb{Z}$ given by xRy if $|x| = |y|$.

(2) The relation R on $\mathbb{Z}$ given by xRy if $2x + y$ is divisible by 3.
(3) The relation R on $\mathbb{Q} \times \mathbb{Q} \setminus \{(0,0)\}$ given by $(a,b)R(c,d)$ if $ad = bc$. Why did we need to remove $(0,0)$ for this to work?

Induction.

Problem 2.5. Do Judson, Ch. 1, Exercises 1 and 3.

Divisibility and Factorization.

Problem 2.6. Do Judson, Ch. 1, Exercises 27 and 31.

Problem 2.7. Let $p_1, \ldots, p_k$ be distinct prime numbers and let $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{Z}_{>0}$ be nonnegative integers. Let $n = p_1^{a_1} \cdots p_k^{a_k}$ and let $m = p_1^{b_1} \cdots p_k^{b_k}$. Compute (with proof!) the prime factorization of $\gcd(n, m)$.

Problem 2.8. Prove or give a counterexample: if $a, b, c \in \mathbb{N}$ such that $\gcd(a,b) = 1$, $a|c$, and $b|c$, then $ab|c$.

3. Creative Problems

Sets and Functions.

Problem 3.1. (Binary Operations) Let A be a set. A *binary operation* on A is a function $f : A \times A \to A$.

(1) We say the binary operation f is *commutative* if $f(a,b) = f(b,a)$ for all $a, b \in A$. Give three examples of commutative binary operations and three examples of non-commutative binary operations.
(2) We say f is *associative* if $f(a, f(b,c)) = f(f(a,b), c)$ for all $a, b, c \in A$. Give an example of an associative binary operation which is not commutative. Give an example of a commutative binary operation which is not associative.
(3) We say f has *left cancellation* if for all $a, b, c \in A$, we have $f(a,b) = f(a,c)$ implies $b = c$. Give an example of a binary operation which does not have left cancellation. Which of your examples above have left cancellation?
(4) Define what it should mean for f to have *right cancellation.* Give an example of a binary operation which has left cancellation but not right cancellation, or else show that left cancellation implies right cancellation.
(5) Dream up your own property P of binary operations. Find as many examples as you can of how your property P relates to the other properties above. (For example, if a binary operation has property P and is commutative, does it have to be associative?) Your property P should be different from the properties chosen by the other people in your group.

Relations.

Problem 3.2. More examples of relations.

(1) Let X be a set and let R be the relation on X given by xRy if $x \neq y$. Show that R is symmetric. Show that if $X \neq \emptyset$, then R is not reflexive. Show that if $|X| \geq 2$, then R is not transitive. (Recall $|X| \geq 2$ means X has at least two different elements.)

(2) Give an example of a relation on $\mathbb{Z}$ which is reflexive and symmetric, but not transitive.

(3) Give an example of a relation on $\mathbb{Z}$ which is symmetric and transitive, but not reflexive.

(4) Give an example of a relation on $\mathbb{Z}$ which is reflexive and transitive, but not symmetric.