

## MATH 113 FINAL REVIEW

### 1. COMPUTATIONS

**Problem 1.1** (Finitely Generated Abelian Groups). Given generators for a subgroup  $H$  of  $\mathbb{Z}^n$ , know how to use matrix manipulation to compute  $\mathbb{Z}^n/H$  as a product of cyclic groups. For example, if  $H$  is the subgroup of  $\mathbb{Z}^4$  generated by  $\{(1, 3, 4, 0), (2, 5, 7, 2), (9, 2, 1, 1)\}$ , you should be able to express  $\mathbb{Z}^4/H$  as a product of cyclic groups.

**Problem 1.2** (Actions). Given a group  $G$  and a  $G$ -set  $X$ , be able to compute the orbits of the action of  $G$  and the stabilizer  $G_x$  for any element  $x \in X$ .

**Examples 1.3.** (1) Let  $G$  be the subgroup  $\{e, r^2, r^4, s, r^2s, r^4s\}$  of  $D_6$ , and let  $G$  act on the vertices of the hexagon in the natural way as a subgroup of  $D_6$ . Find the orbits of the action.

(2) Let  $GL_3(\mathbb{R})$  act on  $\mathbb{R}^3$  in the usual way. Find the orbit and stabilizer of the vector  $(1, 2, 0)$  under this action.

**Problem 1.4** (Burnside's Counting Theorem). Know how to use Burnside's counting theorem to find the number of ways of coloring an object up to a certain symmetry group. For example, you should know how to solve: Suppose that  $\mathbb{Z}/4\mathbb{Z}$  acts on the cube by rotating  $\frac{\pi}{2}$  radians about the center of some chosen face. Find the number of ways of coloring the vertices of the cube with the four colors red, blue, green, and yellow, up to the action of  $\mathbb{Z}/4\mathbb{Z}$ .

**Problem 1.5** (Finite Fields). Know how to construct and do computations in finite fields. For example, be able to construct a field with 27 elements, and know how to multiply elements in this field.

**Problem 1.6** (Polynomials). Know the many ways to show that a polynomial  $p(x) \in \mathbb{Z}[x]$  is irreducible over  $\mathbb{Q}$ ; in particular, know the Gauss lemma, reduction mod  $p$ , and Eisenstein's Criterion. Given a field extension  $E$  of a field  $F$  and an algebraic element  $\alpha \in E$ , be able to find the minimal polynomial of  $\alpha$  over  $F$ .

### 2. THEORY

Of course, you have to know all of the major theorems and results from class and how to use them. Of particular importance are the isomorphism theorems.

**Problem 2.1.** Memorize the proofs of the following theorems from class:

- (1) A subgroup of an  $n$ -generated group is  $n$ -generated.
- (2) The group  $(\mathbb{Z}/p\mathbb{Z})^\times$  is cyclic.
- (3) If  $E$  is a field extension of  $F$  and  $\alpha \in E$  is algebraic, there is a unique irreducible monic polynomial  $p(x) \in F[x]$  such that  $p(\alpha) = 0$ .