Math 113 Final Exam

August 13th, 2009

Name ____________________

Question	Score	Possible
1.1		4
1.2		4
1.3		5
2.1		4
2.2		5
2.3		4
2.4		4
$\sum$		30

1 Computations

Problem 1.1. Take for granted that the group of rigid motions of the tetrahedron is A_4, where the action of $\sigma \in A_4$ is given by permuting the vertices of the tetrahedron. How many different ways can the vertices of the tetrahedron be colored with the three colors red, blue, and green, up to rigid motion?

Problem 1.2. Let H be the subgroup of $\mathbb{Z}^3$ generated by the three elements

$$(10, 6, 10), \quad (8, 4, 4), \quad (2, 2, 6).$$

Compute $\mathbb{Z}^3/H$ as a product of cyclic groups.

Problem 1.3. Let $p(x) = x^4 + x + 1 \in \mathbb{Z}/2\mathbb{Z}[x]$. Do the following:

1. Show that $F = (\mathbb{Z}/2\mathbb{Z}[x])/(p(x))$ is a field.
2. Let $\alpha = \bar{x} \in F$. Find the minimal polynomial of $\alpha^2 + \alpha + 1$ over $\mathbb{Z}/2\mathbb{Z}$.

2 Theory

Problem 2.1. Let $p(x) \in \mathbb{Z}[x]$ be a monic polynomial. Suppose that $p(x) = a(x)b(x)$ where $a(x), b(x) \in \mathbb{Q}[x]$ and $a(x)$ is monic. Show that $a(x), b(x) \in \mathbb{Z}[x]$.

Problem 2.2. Let R be a commutative ring. An element $a \in R$ is called *nilpotent* if there is some $k \in \mathbb{N}$ such that $a^k = 0$.

1. Show that the set of nilpotent elements of R is an ideal.
2. Give an example of a nonzero nilpotent element in the ring $\mathbb{Q}[x]/(x^2 + 6x + 9)$.

Problem 2.3. Let G be a group and let N be a normal subgroup of G. Suppose that $[G : N] = p$ is prime. Show that if H is another subgroup of G and $N \subseteq H$, then $H = N$ or $H = G$.

Problem 2.4. Show that every subgroup of $\mathbb{Z}^n$ is n-generated.