

# A FAMILY OF VARIETIES WITH EXACTLY ONE POINTLESS RATIONAL FIBER

BIANCA VIRAY

ABSTRACT. We construct a concrete example of a 1-parameter family of smooth projective geometrically integral varieties over an open subscheme of  $\mathbb{P}_{\mathbb{Q}}^1$  such that there is exactly one rational fiber with no rational points. This makes explicit a construction of Poonen.

## 1. INTRODUCTION

We construct a family of smooth projective geometrically integral surfaces over an open subscheme of  $\mathbb{P}_{\mathbb{Q}}^1$  with the following curious arithmetic property: there is exactly one  $\mathbb{Q}$ -fiber with no rational points. Our proof makes explicit a non-effective construction of Poonen [Poo09, Prop. 7.2], thus giving “an extreme example of geometry *not* controlling arithmetic” [Poo09, p.2]. We believe that this is the first example of its kind.

**Theorem 1.1.** *Define  $P_0(x) := (x^2 - 2)(3 - x^2)$  and  $P_{\infty}(x) := 2x^4 + 3x^2 - 1$ . Let  $\pi: X \rightarrow \mathbb{P}_{\mathbb{Q}}^1$  be the Châtelet surface bundle over  $\mathbb{P}_{\mathbb{Q}}^1$  given by*

$$y^2 + z^2 = (6u^2 - v^2)^2 P_0(x) + (12v^2)^2 P_{\infty}(x),$$

where  $\pi$  is projection onto  $(u : v)$ . Then  $\pi(X(\mathbb{Q})) = \mathbb{A}_{\mathbb{Q}}^1(\mathbb{Q})$ .

Note that the degenerate fibers of  $\pi$  do not lie over  $\mathbb{P}^1(\mathbb{Q})$  so the family of smooth projective geometrically integral surfaces mentioned above contains all  $\mathbb{Q}$ -fibers.

The non-effectivity in [Poo09, Prop. 7.2] stems from the use of higher genus curves and Faltings’ theorem. (This is described in more detail in [Poo09, §9]). We circumvent the use of higher genus curves by an appropriate choice of  $P_{\infty}(x)$ .

## 2. BACKGROUND

This information can be found in [Poo09, §3,5, and 6]. We review it here for the reader’s convenience.

Let  $\mathcal{E}$  be a rank 3 vector sheaf on a  $k$ -variety  $B$ . A conic bundle  $C$  over  $B$  is the zero locus in  $\mathbb{P}\mathcal{E}$  of a nowhere vanishing zero section  $s \in \Gamma(\mathbb{P}\mathcal{E}, \text{Sym}^2(\mathcal{E}))$ . A diagonal conic bundle is a conic bundle where  $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$  and  $s = s_1 + s_2 + s_3$ ,  $s_i \in \Gamma(\mathbb{P}\mathcal{E}, \mathcal{L}_i^{\otimes 2})$ .

Now let  $\alpha \in k^{\times}$ , and let  $P(x) \in k[x]$  be a separable polynomial of degree 3 or 4. Consider the diagonal conic bundle  $X$  given by  $B = \mathbb{P}^1$ ,  $\mathcal{L}_1 = \mathcal{O}$ ,  $\mathcal{L}_2 = \mathcal{O}$ ,  $\mathcal{L}_3 = \mathcal{O}(2)$ ,  $s_1 = 1$ ,  $s_2 = -\alpha$ ,  $s_3 = -w^4 P(x/w)$ . This smooth conic bundle contains the affine hypersurface  $y^2 - \alpha z^2 = P(x) \subset \mathbb{A}^3$  as an open subscheme. We say that  $X$  is the Châtelet surface given by

$$y^2 - \alpha z^2 = P(x).$$

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Note that since  $P(x)$  is not identically zero,  $X$  is an integral surface.

A Châtelet surface bundle over  $\mathbb{P}^1$  is a flat proper morphism  $V \rightarrow \mathbb{P}^1$  such that the generic fiber is a Châtelet surface. We can construct them in the following way. Let  $P, Q \in k[x, w]$  be linearly independent homogeneous polynomials of degree 4 and let  $\alpha \in k^\times$ . Let  $V$  be the diagonal conic bundle over  $\mathbb{P}_{(a,b)}^1 \times \mathbb{P}_{(w:x)}^1$  given by  $\mathcal{L}_1 = \mathcal{O}, \mathcal{L}_2 = \mathcal{O}, \mathcal{L}_3 = \mathcal{O}(1, 2), s_1 = 1, s_2 = -\alpha, s_3 = -(a^2P + b^2Q)$ . By composing  $V \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  with the projection onto the first factor, we realize  $V$  as a Châtelet surface bundle. We say that  $V$  is the Châtelet surface bundle given by

$$y^2 - \alpha z^2 = a^2P(x) + b^2Q(x),$$

where  $P(x) = P(x, 1)$  and  $Q(x) = Q(x, 1)$ . We can also view  $a, b$  as relatively prime, homogeneous, degree  $d$  polynomials in  $u, v$  by pulling back by a suitable degree  $d$  map  $\phi: \mathbb{P}_{(u:v)}^1 \rightarrow \mathbb{P}_{(a:b)}^1$ .

### 3. PROOF OF THEOREM 1.1

By [Isk71], we know that the Châtelet surface

$$y^2 + z^2 = (x^2 - 2)(3 - x^2)$$

violates the Hasse principle, i.e. it has  $\mathbb{Q}_v$ -rational points for all completions  $v$ , but no  $\mathbb{Q}$ -rational points. Thus,  $\pi(X(\mathbb{Q})) \subseteq \mathbb{A}_{\mathbb{Q}}^1(\mathbb{Q})$ . Therefore, it remains to show that  $X_{(u:1)}$ , the Châtelet surface defined by

$$y^2 + z^2 = (6u^2 - 1)^2P_0(x) + 12^2P_\infty(x),$$

has a rational point for all  $u \in \mathbb{Q}$ .

If  $P_{(u:1)} := (6u^2 - 1)^2P_0(x) + 12^2P_\infty(x)$  is irreducible, then by [CTSSD87a], [CTSSD87b] we know that  $X_{(u:1)}$  satisfies the Hasse principle. Thus it suffices to show that  $P_{(u:1)}$  is irreducible and  $X_{(u:1)}(\mathbb{Q}_v) \neq \emptyset$  for all  $u \in \mathbb{Q}$  and all places  $v$  of  $\mathbb{Q}$ .

**3.1. Irreducibility.** We prove that for any  $u \in \mathbb{Q}$   $P_{(u:1)}(x)$  is irreducible in  $\mathbb{Q}[x]$  by proving the slightly more general statement, that for all  $t \in \mathbb{Q}$

$$P_t(x) := (2x^4 + 3x^2 - 1) + t^2(x^2 - 2)(3 - x^2) = x^4(2 - t^2) + x^2(3 + 5t^2) + (-6t^2 - 1)$$

is irreducible in  $\mathbb{Q}[x]$ . We will use the fact that if  $a, b, c \in \mathbb{Q}$  are such that  $b^2 - 4ac$  and  $ac$  are not squares in  $\mathbb{Q}$  then  $p(x) := ax^4 + bx^2 + c$  is irreducible in  $\mathbb{Q}[x]$ .

Let us first check that for all  $t \in \mathbb{Q}$ ,  $(3 + 5t^2)^2 - 4(2 - t^2)(-6t^2 - 1)$  is not a square in  $\mathbb{Q}$ . This is equivalent to proving that the affine curve  $C: w^2 = t^4 + 74t^2 + 17$  has no rational points. The smooth projective model,  $\overline{C}: w^2 = t^4 + 74t^2s^2 + 17s^4$  in weighted projective space  $\mathbb{P}(1, 1, 2)$ , has 2 rational points at infinity. Therefore  $\overline{C}$  is isomorphic to its Jacobian. A computation in **Magma** shows that  $\text{Jac}(C)(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$  [BCP97]. Therefore, the points at infinity are the only 2 rational points of  $\overline{C}$  and thus  $C$  has no rational points.

Now we will show that  $(-6t^2 - 1)(2 - t^2)$  is not a square in  $\mathbb{Q}$  for any  $t \in \mathbb{Q}$ . As above, this is equivalent to determining whether  $C': w^2 = (-6t^2 - 1)(2 - t^2)$  has a rational point. Since 6 is not a square in  $\mathbb{Q}$ , this is equivalent to determining whether the smooth projective model,  $\overline{C}'$ , has a rational point. The curve  $\overline{C}'$  is a genus 1 curve so it is either isomorphic to its Jacobian or has no rational points. A computation in **Magma** shows that  $\text{Jac}(C')(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$  [BCP97]. Thus  $\#C'(\mathbb{Q}) = 0$  or 2. If  $(t, w)$  is a rational point of  $C'$ , then  $(\pm t, \pm w)$  is

also a rational point. Therefore,  $\#C(\mathbb{Q}) = 2$  if and only if there is a point with  $t = 0$  or  $w = 0$  and one can easily check that this is not the case.

### 3.2. Local Solvability.

**Lemma 3.1.** *For any point  $(u : v) \in \mathbb{P}_{\mathbb{Q}}^1$ , the Châtelet surface  $X_{(u:v)}$  has  $\mathbb{R}$ -points and  $\mathbb{Q}_p$ -points for every prime  $p$ .*

*Proof.* Let  $a = 6u^2 - v^2$  and let  $b = 12v^2$ . We will refer to  $a^2P_0(x) + b^2P_\infty(x)$  both as  $P_{(a:b)}$  and  $P_{(u:v)}$ .

$\mathbb{R}$ -points: It suffices to show that given  $(u : v)$  there exists an  $x$  such that

$$P_{(a:b)} = x^4(2b^2 - a^2) + x^2(3b^2 + 5a^2) + (-6a^2 - b^2)$$

is positive. If  $2b^2 - a^2$  is positive, then any  $x$  sufficiently large will work. So assume  $2b^2 - a^2$  is negative. Then  $\alpha = \frac{-(3b^2 + 5a^2)}{2(2b^2 - a^2)}$  is positive. We claim  $P_{(a:b)}(\sqrt{\alpha})$  is positive.

$$\begin{aligned} P_{(a:b)}(\sqrt{\alpha}) &= \alpha^2(2b^2 - a^2) + \alpha(3b^2 + 5a^2) + (-6a^2 - b^2) \\ &= \frac{(3b^2 + 5a^2)^2}{4(2b^2 - a^2)} + \frac{-(3b^2 + 5a^2)^2}{2(2b^2 - a^2)} + (-6a^2 - b^2) \\ &= \frac{1}{4(2b^2 - a^2)} (4(2b^2 - a^2)(-6a^2 - b^2) - (3b^2 + 5a^2)^2) \\ &= \frac{1}{4(2b^2 - a^2)} (-17b^4 - 74a^2b^2 - a^4) \end{aligned}$$

Since  $2b^2 - a^2$  is negative by assumption and  $-17b^4 - 74a^2b^2 - a^4$  is always negative, we have our result.

$\mathbb{Q}_p$ -points:

$p \geq 5$ : Without loss of generality, let  $a$  and  $b$  be relatively prime integers. Let  $\overline{X}_{(a:b)}$  denote the reduction of  $X_{(a:b)}$  modulo  $p$ . We claim that there exists a smooth  $\mathbb{F}_p$ -point of  $\overline{X}_{(a:b)}$  that, by Hensel's lemma, we can lift to a  $\mathbb{Q}_p$ -point of  $X_{(a:b)}$ .

Since  $P_{(a:b)}$  has degree at most 4 and is not identically zero modulo  $p$ , there is some  $x \in \mathbb{F}_p$  such that  $P_{(a:b)}(x)$  is nonzero. Now let  $y, z$  run over all values in  $\mathbb{F}_p$ . Then the polynomials  $y^2, P_{(a:b)}(x) - z^2$  each take  $(p+1)/2$  distinct values. By the pigeonhole principle,  $y^2$  and  $P_{(a:b)}(x) - z^2$  must agree for at least one pair  $(y, z) \in \mathbb{F}_p^2$  and one can check that this pair is not  $(0, 0)$ . Thus, this tuple  $(x, y, z)$  gives a smooth  $\mathbb{F}_p$ -point of  $\overline{X}_{(a:b)}$ . (The proof above that the quadratic form  $y^2 + z^2$  represents any element in  $\mathbb{F}_p$  is not new. For example, it can be found in [Coh07, Prop 5.2.1].)

$p = 3$ : From the equations for  $a$  and  $b$ , one can check that for any  $(u : v) \in \mathbb{P}_{\mathbb{Q}}^1$ ,  $v_3(b/a)$  is positive. Since  $\mathbb{Q}_3(\sqrt{-1})/\mathbb{Q}_3$  is an unramified extension, it suffices to show that given  $a, b$  as above, there exists an  $x$  such that  $P_{(a:b)}(x)$  has even valuation. Since  $v_3(b/a)$  is positive,  $v_3(2b^2 - a^2) = 2v_3(a)$ . Therefore, if  $x = 3^{-n}$ , for  $n$  sufficiently large, the valuation of  $P_{(a:b)}(x)$  is  $-4n + 2v_3(a)$  which is even.

$p = 2$ : From the equations for  $a$  and  $b$ , one can check that for any  $(u : v) \in \mathbb{P}_{\mathbb{Q}}^1$ ,  $v_2(b/a)$  is at least 2. Let  $x = 0$  and  $y = a$ . Then we need to find a solution to  $z^2 = a^2(-7 + (b/a)^2)$ . Since  $v_2(b/a) > 1$ ,  $-7 + (b/a)^2 \equiv 1^2 \pmod{8}$ . By Hensel's lemma, we can lift this to a solution in  $\mathbb{Q}_2$ . □

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720-3840, USA  
E-mail address: [bviray@math.berkeley.edu](mailto:bviray@math.berkeley.edu)  
URL: <http://math.berkeley.edu/~bviray>