# Worksheet 17: Linear Approximation & Differentials

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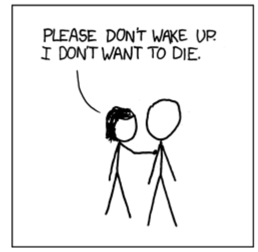


www.xkcd.com

1. Find the linearization of $f(x) = \sin(x)$ at $a = \frac{\pi}{2}$.
   Taking the derivative, $f'(x) = \cos(x)$, so $f'(a) = f'(\frac{\pi}{2}) = 0$. Thus the linearizion (tangent line at $a$) is $L(x) = f(a) + 0(x - a) = \sin(\frac{\pi}{2}) + 0 = 1$.

2. Find the linearization of $f(x) = x^{\frac{3}{4}}$ at $a = 16$.
   Taking the derivative, $f'(x) = \frac{3}{4}x^{-\frac{1}{4}}$, so $f'(a) = \frac{3}{4}(16)^{-\frac{1}{4}} = \frac{3}{8}$.

   Thus the linearizion (tangent line at $a$) is

$$\begin{aligned} L(x) &= f(a) + \frac{3}{2}(x - a) \\ &= (16)^{\frac{3}{4}} + \frac{3}{2}(x - 16) \\ &= 8 + \frac{3}{2}(x) - 24 \\ &= \frac{3}{2}(x) - 16 \end{aligned}$$

3. Find the differential of each function:

   (a) $y = \frac{t^2}{1-t^5}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{(1-t^5)(2t) - t^2(-5t^4)}{(1-t^5)^2} \\ &= \frac{2t - 2t^6 + 5t^6}{(1-t^5)^2} \\ &= \frac{2t + 3t^6}{(1-t^5)^2} \\ dy &= \left(\frac{2t + 3t^6}{(1-t^5)^2}\right) dx \end{aligned}$$

   (b) $y = e^x \cos^{-1}(x^2)$

$$\begin{aligned} \frac{dy}{dx} &= e^x \cos^{-1}(x^2) + \frac{-2xe^x}{\sqrt{1-x^4}} \\ dy &= \left(e^x \cos^{-1}(x^2) + \frac{-2xe^x}{\sqrt{1-x^4}}\right) dx \end{aligned}$$

4. Use a linear approximation or differentials to estimate the given number:

   (a) $e^{-.01}$

   Note that the function under consideration is $f(x) = e^x$ and $a = 0$ is the obvious choice. Solving by differentials, $dy = e^x dx$. So, setting $x = a$ and $dx = -.01$, $dy = e^0(-.01) = -.01$. The estimate of $f(-.01)$ is therefore $f(0) + (-.01) = 1 - .01 = .99$.

(b) $\frac{1}{4.001}$

Note that the function under consideration is $f(x) = \frac{1}{x}$ and $a = 4$ is the obvious choice. Solving by linearization, $f'(x) = -x^{-2}$, and so $f'(4) = -\frac{1}{16}$. Thus,

$$\begin{aligned} L(x) &= f(a) - \frac{1}{16}(x-a) \\ &= \frac{1}{4} - \frac{1}{16}(x-4) \end{aligned}$$

And so, setting $x = 4.001$,

$$\begin{aligned} L(4.001) &= \frac{1}{4} - \frac{1}{16}(4.001-4) \\ &= \frac{1}{4} - \frac{1}{16}(\frac{1}{1000}) \\ &= \frac{1}{4} - \frac{1}{16000} \\ &= \frac{4000}{16000} - \frac{1}{16000} \\ &= \frac{3999}{16000} \end{aligned}$$

(c) $\sqrt{99.9}$

Note that the function under consideration is $f(x) = \sqrt{x}$ and $a = 100$ is the obvious choice. Solving by differentials, $dy = \frac{1}{2}x^{-\frac{1}{2}}dx$. So, setting $x = a$ and $dx = -.1$,

$$\begin{aligned} dy &= \frac{1}{2}(100)^{-\frac{1}{2}}(-.1) \\ &= \frac{1}{2}(\frac{1}{10})(-.1) \\ &= (\frac{1}{20})(-.1) \\ &= -\frac{1}{200} \end{aligned}$$

The estimate of $f(99.9)$ is therefore $f(100) + (-\frac{1}{200}) = 10 - .005 = 9.995$

5. The radius of circular disk is given as 24 cm with a maximum error in measurement of .2 cm. Use differentials to calculate the maximum error in the calculated area of the disk; what is the relative error?

   Note that the formula for the area of a circle is: $A = \pi r^2$. Taking the derivative with respect to radius (note that the error in measurement is a change in radius), $\frac{dA}{dr} = 2\pi r$, and so the relevant differential is $dA = 2\pi r dr$. Plugging in the values, given we obtain that the maximum error is $dA = 2\pi(24)(.2) = \frac{48\pi}{5}$. The error relative to the area, then, is $\frac{\frac{48\pi}{5}}{(24)^2\pi} = \frac{1}{60}$.

6. ($\star$) Is there any difference between the approximation given by a differential and the approximation given by a linearization? Why or why not?

   No; they're both using the same tangent line. It's two different ways of looking at the same approximation.