Chip Firing on Graphs

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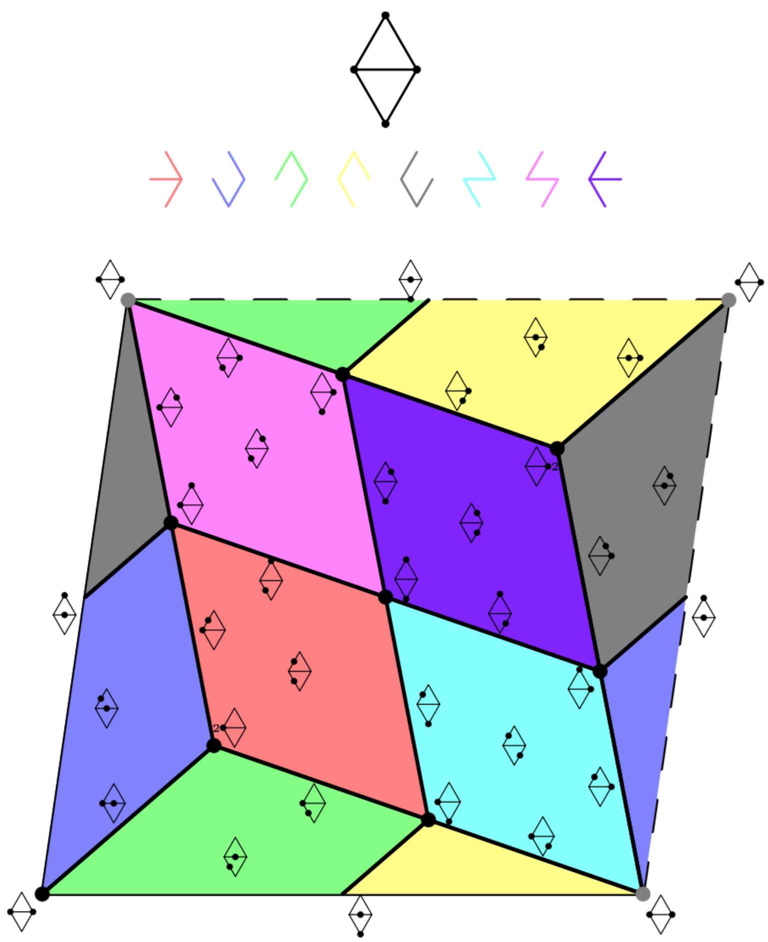
November 17, 2015

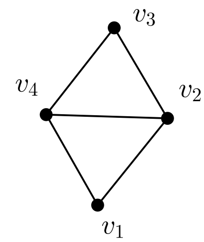
1 Divisors

Our goal for this talk is to understand the following picture, which comes from the paper *Canonical Reresentatives for Divisor Classes on Tropical Curves and the Matrix-Tree Theorem* by An, Baker, Kuperberg, and Shokrieh.



Definition 1. A *graph* G is a pair (V, E), where V is any set, and E is any collection of pairs from V. We call elements of V the *vertices* of the graph, and the elements E are the of *edges* of the graph. The number of edges containing a vertex $v \in V$ is called the *degree* of v.

Example We usually represent graphs by drawing them. Here, the set $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{\{v_1, v_2\}, \{v_2, v_4\}, ...\}$. The degree of v_1 is two and the degree of v_2 is 3.

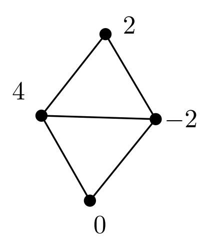


We will now describe a game that can be played on graphs. Think of the vertices in V as individuals, and the edges E as relationships between them, and the entire graph G as a community. Each of the people has a certain amount of money, which we will describe using *divisors*. There will also be a way for community members to exchange money.

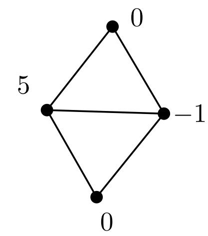
Definition 2. A *divisor* is a function $D : V \to \mathbb{Z}$. We think of D as "assigning" a number of chips to the vertex v. If you like, a divisor is an element of the free abelian group on the vertices. The *degree* of the divisor is the total amount of money in the community, or

$$\sum_{v \in V} D(v).$$

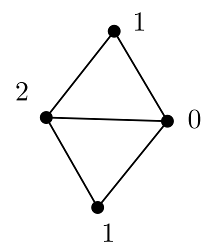
Example Here, we have a divisor where $D(v_1) = 0$, $D(v_2) = -2$, and so on. The degree of this divisor is 4.



A vertex in this graph may decide to *lend* or *borrow*. If a vertex v makes a lending move, we define a new divisor where v loses degree of v many chips, and each vertex connected to v gains one chip. Similarly, if v makes a borrowing move, we define a new divisor where v gains degree of v many chips, and each vertex connected to v loses one chip. For example, with the above divisor, if v_3 makes a lending move, then we have



Our goal is to get all vertices out of debt. We can fire at v_4 to obtain the following divisor:



If a sequence of lending and borrowing moves exists which brings a divisor D out of debt, we say that D is *winnable*.

Question What are some examples of divisors which are not winnable on this graph?

As mathematicians, we want a systematic way for deciding whether or not a given divisor is winnable. We notice that a divisor D is winnable iff D' is winnable, where D' is any divisor which can be obtained from D by a sequence of lending and borrowing moves. This means that the question we are asking might be easier to answer if, instead of looking at individual divisors, we look at collections of divisors which are related to each other via chip firing. To that end, we will define *divisor classes*.

Definition 3. We will say two divisors are *equivalent* if we can get from one to another via a sequence of lending or borrowing moves.

Question Let's check that this is an equivalence relation. (symmetric, reflexive, transitive)

Definition 4. The *divisor class* determined by a divisor D is

$$[D] = \{D' \mid D' \text{ is equivalent to } D\}.$$

So, this is the set of all divisors which we can get starting from D, using lending and borrowing moves.

Now, we may rephrase our goal of getting everyone out of debt: given a divisor D, is there en element of $[D]$ which is out of debt?

Definition 5. The *Picard Group* Pic(G) is the set of divisor classes on G:

$$\mathrm{Pic}(G) = \{[D] \mid D \text{ is a divisor on } G\}$$

Let $\mathrm{Pic}^d(G)$ be the set of divisor classes of degree d on G.

Executive Session: We can put a group structure on Pic by adding divisors vertex-wise. You can also show this operation is well defined with respect to the equivalence. Then, we have the following isomorphism. Fix $q \in G$, and let $\deg(D)_q$ be the divisor with degree D chips on the vertex q. Then

$$\mathrm{Pic}(G) \to \mathbb{Z} \times \mathrm{Pic}^0(G)$$

$$[D] \mapsto (\deg(D), [D - \deg(D)_q])$$

is an isomorphism of groups. So, if we figure out canonical representatives for divisor classes in $\mathrm{Pic}^n(G)$ for some n, then we know what happens in all n.

2 Metric Graphs

Definition 6. A *metric graph* Γ is a graph in which the edges have lengths, and every point (including points along "edges") are vertices. A *divisor* D on Γ is a function $D : \Gamma \to \mathbb{Z}$, such that the degree is finite. We think of this as an assignment of different numbers of chips on Γ.

We define firing moves in the same way, except that this time we must also specify a distance ϵ for the vertex to fire. This distance must be small enough so that it does not cause any problems. We also have a notion of equivalence in this setting, and the picard group $\mathrm{Pic}(\Gamma)$.

Definition 7. A *spanning tree* of a graph is a minimal subset of the edge set which covers all of the vertices. In other words, it is a subgraph of G which hits every vertex but contains no cycles.

Definition 8. A *break divisor* on Γ is a divisor which is supported on the closure of the complement of a spanning tree in Γ. All break divisors have the same degree, and we call this g.

We will now discuss several theorems that allow us to picture the divisor classes on Γ very easily.

Theorem 1 (Mikhalkin and Zharkov)**.** *Each element of $\mathrm{Pic}^g(\Gamma)$ can be uniquely represented by a break divisor.*

Theorem 2 (Mikhalkin and Zharkov)**.** *We can picture $Pic^g(\Gamma)$ as a real torus of dimension g, where g is as above.*

Given any spanning tree T of G let Σ_T be the break divisors supported on $\Gamma \backslash T$. Any break divisor uniquely determines a spanning tree T, so for two distinct spanning trees T, T', we have that $\Sigma_T \cap \Sigma_{T'} = \emptyset$. Then the mapping sending $\mathrm{Pic}^g(\Gamma)$ to $J(\Gamma)$ maps Σ_T to an open paralleletope. These parallelotopes give the *canonical cell decomposition of* $J(G)$.

3 Conclusion and Discussion of the Figure

Each element of $\mathrm{Pic}^g(\Gamma)$ is represented by a unique break divisor, and all break divisors are represented in the picture. Each parallelogram corresponds to one spanning tree in Γ. Shifting a chip along one edge of the graph moves you parallel to one side of this parallelogram, while moving the other chip moves you in the other direction. The sides of the parallelograms correspond to when one chip is on a vertex, and the vertices in the picture correspond to divisor classes on the graph G.

Using the *executive session*, we know that given any divisor of degree d, we can fix a vertex q and subtract $d - g$ chips from q, and the resulting divisor will have a unique representative as a break divisor.