LIMIT OF VORONOI AND DELAUNAY CELLS: MPI LEIPZIG

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This project began with ice cream (see youtube).

1. Voronoi and Delaunay Cells of Varieties

Throughout let $X \subset \mathbb{R}^2$ be a real plane curve:

$$X = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\},$$

where $f(x, y)$ is a polynomial equation.

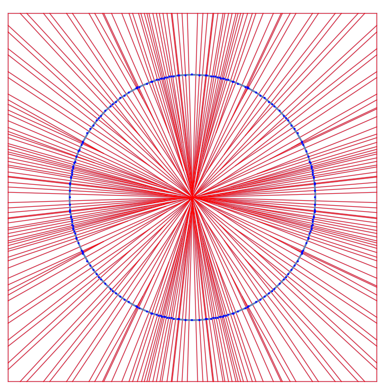
Definition 1.1. Given a point $x \in X$, define the *Voronoi cell of* x to be

$$\mathrm{Vor}_X(x) = \{y \in \mathbb{R}^2 \mid d(y, x) \leq d(y, x') \text{ for all } x' \in X\}.$$

This is a convex semialgebraic set of dimension 1 so long as x is a smooth point of X. It is contained in the *normal space to* X *at* x:

$$N_X(x) = \{u \in \mathbb{R}^n \mid u - x \text{ is perpendicular to the tangent space of } X \text{ at } x\}.$$

Example 1.2. (Audience participation: Voronoi cells of a circle)



Example 1.3. (Audience participation: Voronoi cells of an elliptic curve) Consider the curve defined by $x(x - 0.5)(x + 0.5) - y^2 = 0$.

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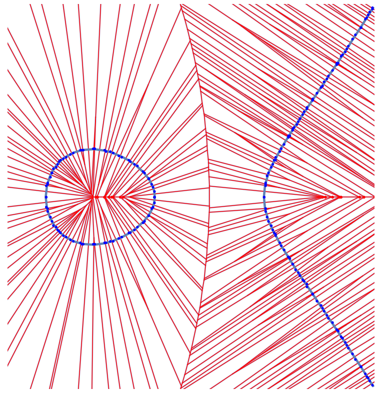
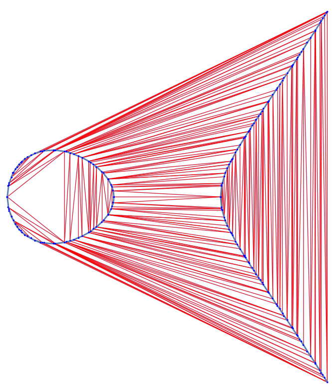


Let $B(p, r)$ denote the open disc with center $p \in \mathbb{R}^n$ and radius $r > 0$. We say this disc is *inscribed* with respect to X if $X \cap B(p, r) = \emptyset$ and we say it is *maximal* if no disc containing $B(p, r)$ shares this property. Each inscribed sphere in X gives a Delaunay cell, defined as follows.

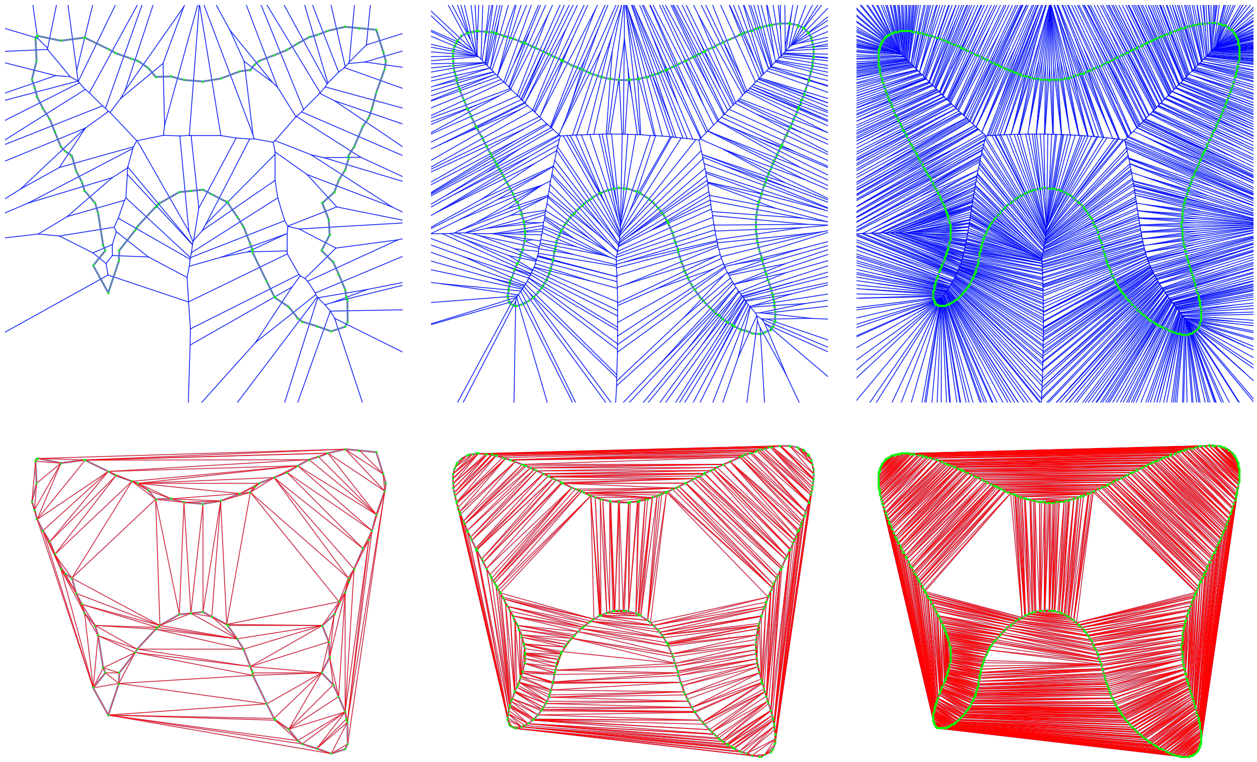
Definition 1.4. Given an inscribed sphere B of a variety $X \subset \mathbb{R}^n$, the *Delaunay cell* $Del_X(B)$ is $conv(\overline{B} \cap S)$.

Example 1.5. (Audience participation: Delaunay cells of an elliptic curve)



2. Limits of Voronoi and Delaunay Cells of Curves

We now wish to study convergence of Voronoi and Delaunay cells. More precisely, given a real algebraic curve X and a sequence of samplings $A_N \subset X$ with $|A_N| = N$, we wish to show that Voronoi (or Delaunay) cells from the Voronoi (or Delaunay) diagrams of the A_N limit to Voronoi (or Delaunay) cells of X. We begin by introducing two notions of convergence which we will use to describe the limits.



The *Hausdorff distance* of two compact sets B_1 and B_2 in $\mathbb{R}^n$ is defined as

$$d_h(B_1, B_2) = \sup\left\{\sup_{x\in B_1}\inf_{y\in B_2} d(x,y), \sup_{y\in B_2}\inf_{x\in B_1} d(x,y)\right\}.$$

A sequence $\{B_\nu\}_{\nu\in\mathbb{N}}$ of compact sets is *Hausdorff convergent* to B if $d_h(B, B_\nu) \to 0$ as $\nu \to \infty$. Given a point $x \in \mathbb{R}^n$ and a closed set $B \subset \mathbb{R}^n$, let

$$d_w(x, B_1) = \inf_{b\in B} d(x, b).$$

Given a sequence of closed sets B_i, we say they are *Wijsman convergent* to B if for every $x \in \mathbb{R}^n$, we have that

$$d_w(x, A_i) \to d_w(x, A).$$

We use Wijsman convergence as a variation of Hausdorff convergence which is well suited for unbounded sets.

An ϵ-*approximation* of a real algebraic variety X is a discrete subset $A_\epsilon \subset X$ such that for all $y \in X$ there exists an $x \in A_\epsilon$ so that $d(y, x) \leq \epsilon$. We need these two notions of convergence because Delaunay cells are always compact, while Voronoi cells may be unbounded. (Give example of wedges).

We now study convergence of Delaunay cells of X, and introduce a condition on real algebraic varieties which ensures that the Delaunay cells are simplices.

Definition 2.1. We say that a real algebraic variety $X \subset \mathbb{R}^n$ is *Delaunay-generic* if X does not meet any d-dimensional inscribed sphere greater than $d + 2$ points.

Example 2.2. (Audience participation) Can anyone provide an example of a variety that is not Deluanay-generic?

Theorem 2.3 (B-Weinstein)**.**

(1) Let $X \subset \mathbb{R}^n$ be a Delaunay-generic real algebraic curve, and let A_ϵ be a sequence of ϵ-approximations of X. Every maximal Delaunay cell is the Hausdorff limit of a sequence of Delaunay cells of A_ϵ.

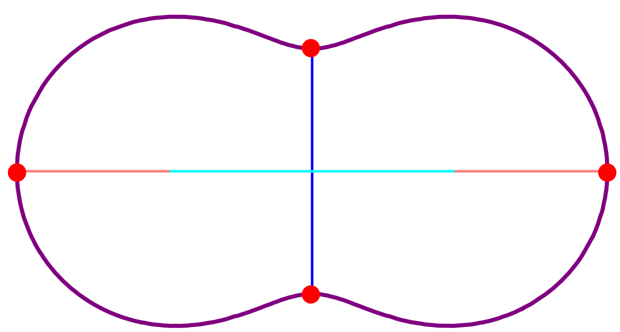
(2) Let X be a compact and smooth curve in $\mathbb{R}^2$ and $\{A_\epsilon\}_{\epsilon \searrow 0}$ be a sequence of ϵ-approximations. Then every Voronoi cell is the Wijsman limit of a sequence of Voronoi cells of A_ϵ.

Right now we are working on the singular case for Voronoi cells.

3. Metric Features of Plane Curves

Curvature, bottlenecks, medial axis, reach.

Example 3.1. Consider the plane curve $((x-1)^2+y^2-1)((x+1)^2+y^2-1)-1/2$.



It has two points of maximal curvature, $(-2.03, 0)$ and $(2.03, 0)$. The radius of curvature at these points is 1.04 and is shown in pink. This means that balls centered at $(\pm 0.99, 0)$ with radius 1.04 are maximally inscribed, as before.

The narrowest bottleneck is between the points $(0, -0.84)$ and $(0, 0.84)$. A bottleneck is a pair of points on the curve who are contained in one an other's normal line.

The medial axis is shown in light blue. The medial axis of the curve is all points which have more than one nearest point on the curve.

The reach is the smallest distance from the curve to its medial axis. It is also the minimum of the minimal radius of curvature and half the narrowest bottleneck. Therefore, the reach of this curve is 0.84.

So, in the rest of the paper, we look at each of these metric features and relate them to the Voronoi cells of the curve, and then obtain a convergence result which tells you how to see these features from the Voronoi cells of an ϵ-approximation of the curve.

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