Tropicalization of Genus 2 Curves

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1 Goal

In this project, we study the tropicaliztion of genus 2 curves. Given a genus 2 curve C over a valued field K, there is a genus 2 tropical curve we can associate to C which is unique up to isomorphism of curves. It is a nontrivial task to compute the tropical curve associated to C, and the goal of this talk is to understand how to compute this tropical curve in two different ways:

1. By studying roots in the hyperelliptic cover, [RSS14]

2. By studying the Igusa Invariants. [Hel]

2 From curves to tropical curves

Setup: In what follows K will be a field with valuation with maximal ideal m and valuation ring R, k will be the residue field, and C a curve over K.

We have that there exists a coarse moduli space $\overline{M}_g$ of stable curves. This is the *stable compactification* of M_g. The space $\overline{M}_g$ is compact and separated.

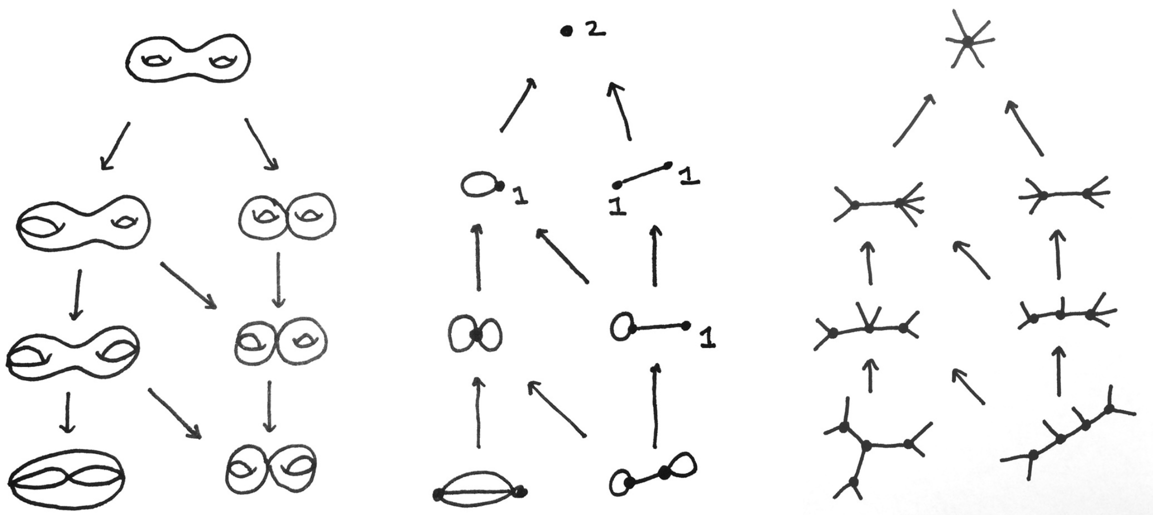
What does this mean intuitively? It means that if we have a geometrically smooth family of curves which specializes to a nonstable curve (having a cusp or worse singularity), we are guaranteed (abstractly) that we can replace the unstable fibers with stable ones by making a base change and binational modification.

Example The curve

$$y^2 = (x-1)(x-2)(x-3)(x-4)(x-t)(x-t^2)$$

defines a hyperelliptic curve over the Puiseux series $\mathbb{C}\{\{t\}\}$. For $t \neq 0$, this defines a smooth curve over $\mathbb{C}$. However, for $t = 0$, (i.e., the fiber over the residue field) we do not have a stable curve. The semistable reduction theorem tells us that we should be able to replace this fiber *uniquely* by some element of $\overline{M}_g$. How can we tell what the reduction type is, and how do we know what the corresponding tropical curve is?

The *reduction type* of the curve completely determines what the associated tropical curve will be, since the tropical curve is the dual graph of the stable curve. There are seven possible reduction types for a genus two curve.



3 Moduli space of genus 2 tropical curves

For our purposes, a *tropical curve* will be a triple (Γ, w, l) where $\Gamma = (V, E)$ is a connected graph, and w is a function from $V \to \mathbb{Z}_{\geq 0}$ assigning weights to the vertices, and l is a function from $E \to \mathbb{R}_{\geq 0}$ assigning a length to each edge.

The *genus* of a tropical curve is the sum over all weights of the vertices, plus the classical genus of the graph Γ.

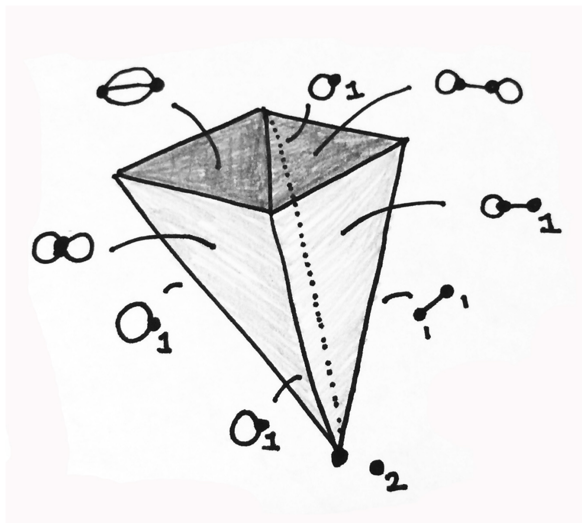
We will say that two tropical curves are isomorphic if one can be obtained from the other via the following operations:

1. Graph automorphisms
2. Removing a leaf of weight 0, together with the edge connected to it.
3. Removing a vertex of degree 2 and weight 0, and replacing the corresponding edges by one edge whose length is the sum of the lengths of the old edges.

4. Removing an edge of length 0 and adding the weights of the corresponding vertices.

In this way, every tropical curve has a *minimal skeleton*. This is a tropical curve with no vertices of weight 0 and degree less than or equal to two, or edges of length zero.

Given a fixed pair (Γ, w), or *combinatorial type*, the moduli space of tropical curves of this type is $\mathbb{R}_{\geq 0}^{|E|}/\mathrm{Aut}(\Gamma)$. The coordinates in $\mathbb{R}_{\geq 0}^{|E|}$ give the edge lengths in the graph. The boundary of these cones corresponds to curves with at least one edge of length 0. Then, we glue the cones along the boundaries to form M_g^{tr}. The moduli space M_g^{tr} is a *stacky fan*, and has been well studied.



4 The two methods

4.1 Method One: Ren, Sam, Sturmfels

Our goal is to understand the map

$$M_2 \to M_2^{tr}.$$

We now discuss the methods presented in [RSS14] for carrying out this map.

Theorem 1 ([RSS14]). *There is a commutative diagram*

$$\begin{array}{ccc} M_{0,6} & \longrightarrow & M_{0,6}^{tr} \\ \downarrow & & \downarrow \\ M_2 & \longrightarrow & M_2^{tr} \end{array}$$

Now, we may study the bottom map by instead studying the top horizontal map and the right vertical map, which we understand well.

The map from

$$M_{0,6} \to M_2$$

takes the genus 2 curve coming from the hyperelliptic cover of $\mathbb{P}^1$ with the 6 marked ramification points.

All genus 2 curves are hyperelliptic, meaning any one can be defined by giving 6 points in $\mathbb{P}^1$. The curve is then the double cover of $\mathbb{P}^1$ branched at the 6 points. If the curve is given in the form

$$y^2 = f(x)$$

For a polynomial f of degree 5 or 6, then the points are precisely the roots of f, plus possibly the point at infinity depending upon if f is degree 5.

The space $M_{0,6}^{tr}$ is one we know well: this is the space of trees with 6 taxa. Using the distances d_{ij} between the 6 marked points, we get a fan in $\mathbb{TP}^{14}$. Combinatorially, it agrees with the tropical Grassmannian $trop(Gr(2,6))$, as described in [MS15]. We know that $M_{0,6}^{tr}$ has a tropical basis given by the Plücker relations for $Gr(2,6)$. It has one 0 dimensional cone, 25 rays, 105 two dimensional cones, and 105 three dimensional cones. The dimension corresponds to the number of interior edges in the tree. Then the map

$$M_{0,6} \to M_{0,6}^{tr}$$

can be described as follows. Denote the 6 points in $\mathbb{P}^1$ by (a_i, b_i). Then, take the matrix

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_6 \\ b_1 & b_2 & \cdots & b_6 \end{bmatrix}$$

and take all 2×2 minors of the matrix. Then, take valuations of each of these. This describes a point in $\mathbb{P}^{14}$

$$\Delta = (p_{12}, p_{13}, \cdots, p_{56})$$

This describes a tree metric on a tree with 6 taxa by taking $d(i,j) = -2p_{ij} + \mathbf{1}$ for a suitable constant n.

The map

$$M_{0,6}^{tr} \to M_2^{tr}$$

This is a morphism of generalized cone complexes, and can be described as follows. Given a point in $M_{0,6}^{tr}$, find the tree associated to it, with interior edge lengths. This tree maps to the corresponding tropical curve, as in the figures. If an interior edge of length l in the tree comes from a curve of type (3), then the loop has length $2l$. Otherwise, if the curve is of type (2), the edge gets length $1/2l$.

4.2 Method Two: Helminck

In [Hel], Helminck completely determines the semistable reduction type of a genus two curve using the *Igusa invariants*. These were first defined in [Igu60], in order to try and extend the knowledge of genus 1 curves obtained by the j-invariant to higher genus.

Definition 1. The *tropical Igusa invariants* are the valuations of J_i and I_i.

He gives the following theorem:

Theorem 2 ([Hel])**.** *Let C be a semistable curve of genus 2 over K. Then the cycle lengths and reduction type of a faithful tropicalization can be completely described in terms of the tropical Igusa invariants.*

Using the tropical Igusa invariants, Helminck can determine which of the seven possible reduction types $\mathcal{C}_S$. The full theorem statement may be found in [Hel], and we implemented this in *Mathematica*. Helminck also determines the *thickness* of the singular points on $\mathcal{C}_S$, which gives us the lengths of the edges in the tropical curve.

5 Examples

5.1 Example

Suppose exactly two points coincide in the residue field. For instance, this is the polynomial introduced in the introduction. Call these two points a_5, a_6. Then the tropical Igusa invariants are

$$(0, 0, 0, 0, 2a, 0, 0, 0, 0, 0),$$

where $a = v(a_5 - a_6) > 0$. This tells us that the curve is irreducible with one singular point of thickness $2a$. This corresponds to a single loop of length 2a and a vertex of weight 1.

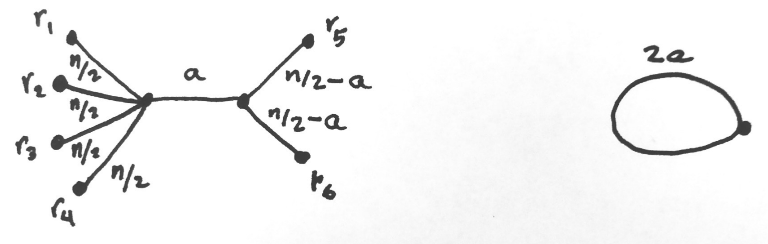
On the other hand, in $M_{0,6}^{tr}$ we have a point of the form

$$(0, \dots, 0, a) = (p_{1,2}, p_{1,3}, \dots, p_{1,5}).$$

This gives us a tree metric

$$(n, \dots, n, n - 2a) = (d_{1,2}, d_{1,3}, \dots, d_{1,5}).$$

So the tree has the desired type, with an interior edge length of a. Then, we double this length to find the length of the corresponding loop in the tropical curve.



5.2 A second example

Consider the polynomial

$$y^2 = (x+1)(x-t)(x-t^2)(x-t^3)(x-t^4).$$

Its tropical Igusa invariants are

$$(2, 4, 6, 8, 20, 2, 6, 6, 8, 20).$$

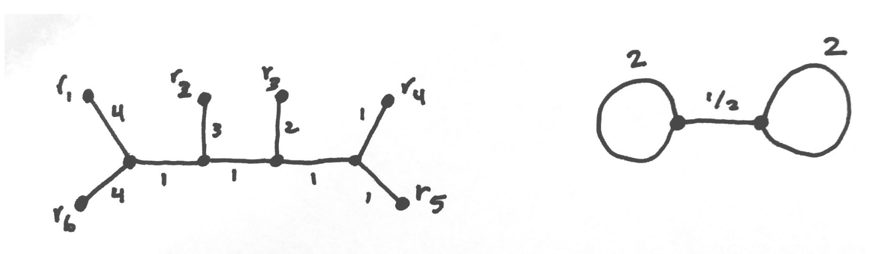
This tells us that the reduction type is two projective lines intersecting in one point, with thicknesses $(1/2, 2, 2)$, so the graph should be the dumbell with each loop having length 2 and the interior edge having length $1/2$.

On the other hand, in $M_{0,6}^{tr}$ we have the point

$$(0, 0, 0, 0, 0, 1, 1, 1, 0, 2, 2, 0, 3, 0, 0) = (p_{1,2}, p_{1,3}, \ldots, p_{1,5}).$$

So that a possible tree metric (up to all 1's vector) is

$$(8, 8, 8, 8, 8, 6, 6, 6, 8, 4, 4, 8, 2, 8, 8) = (d_{1,2}, d_{1,3}, \ldots, d_{1,5}).$$



References

[Abr13] Dan Abramovich. Moduli of algebraic and tropical curves. In *Colloquium de Giorgi 2010–2012*, volume 4 of *Colloquia*, pages 35–47. Ed. Norm., Pisa, 2013.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Hel] Paul Alexander Helminick. Tropical igusa invariants and torision embeddings.

[HM98] Joe Harris and Ian Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[Igu60] Jun-ichi Igusa. Arithmetic variety of moduli for genus two. *Ann. of Math. (2)*, 72:612–649, 1960.

[MS15] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry*, volume 161 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015.

[Pay09] Sam Payne. Analytification is the limit of all tropicalizations. *Math. Res. Lett.*, 16(3):543–556, 2009.

[RSS14] Qingchun Ren, Steven V. Sam, and Bernd Sturmfels. Tropicalization of classical moduli spaces. *Math. Comput. Sci.*, 8(2):119–145, 2014.