Toric Varieties

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Last week we saw that we can define normal toric varieties from the data of a fan in a lattice. Today I will review this idea, and also explain how they can arise from multigraded polynomial rings. The idea is that we can view a toric variety as an open subspace of $\mathbb{C}^n$ modulo an abelian group action.

Example. Throughout the talk I will return to the example of $\mathbb{P}^1 \times \mathbb{P}^1$. This can be thought of as points in $\mathbb{C}^4$ where the first two coordinates do not simultaneously vanish, and the second coordinates do not simultaneously vanish. Points are defined up to simultaneously scaling the first two coordinates or simultaneously scaling the second two coordinates.

1 Recap: Multigradings

Recall from Leon's talk that a multigrading on a polynomial ring $S = \mathbb{C}[x_1, \dots, x_n]$ is specified by an exact sequence of additive abelian groups:

$$0 \to L \to \mathbb{Z}^n \to A \to 0. \tag{1}$$

A monomial $m = x_1^{a_1} \cdots x_n^{a_n} \in S$ is identified with the vector $(a_1, \dots, a_n) \in \mathbb{Z}^n$, and the **degree** of the monomial m is defined to be the image of $(a_1, \dots, a_n)$ in A, where A can be any abelian group.

Example. Let $L = \mathrm{Span}_{\mathbb{Z}}((1, -1, 0, 0), (0, 0, 1, -1))$. In this case, $n = 4$, and $A = \mathbb{Z}^4/L \approx \mathbb{Z}^2$. We identify A with $\mathbb{Z}^2$ by $(a, b, c, d) \mapsto (a + b, c + d)$. The degrees of the variables $x_0, x_1, y_0, y_1 \in S = \mathbb{C}[x_0, x_1, y_0, y_1]$ are

$$\deg(x_0) = (1, 0), \quad \deg(x_1) = (1, 0), \quad \deg(y_0) = (0, 1), \quad \deg(y_1) = (0, 1).$$

2 Recap: Toric Varieties via Glueing

Last week in Chris' lecture, we defined toric varieties and described how to construct normal toric varieties from the data of a fan in a lattice. We will review this idea now, continuing with the example of $\mathbb{P}^1 \times \mathbb{P}^1$.

Definition 1. A **toric variety** X is a variety over $\mathbb{C}$ with a torus $(\mathbb{C}^*)^k$ as an open, dense subset such that group action of the torus $(\mathbb{C}^*)^n$ naturally extends to the entire toric variety X.

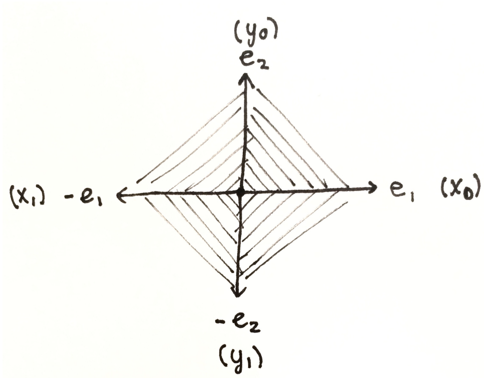
Example. The toric variety $\mathbb{P}^1 \times \mathbb{P}^1$ has coordinates $(x_0 : x_1, y_0 : y_1)$. The toric variety $\mathbb{P}^1 \times \mathbb{P}^1$ has $(\mathbb{C}^*)^2$ as an open, dense subset: $(s,t) \mapsto (s : 1, t : 1)$. The action of the torus $(\mathbb{C}^*)^2$ on $\mathbb{P}^1 \times \mathbb{P}^1$ is given by $(s,t) \cdot (x_0 : x_1, y_0 : y_1) = (sx_0 : x_1, ty_0 : y_1)$.

To construct normal toric varieties, we first defined affine toric varieties as the spectra of semigroup rings. The semigroup rings are specified by cones inside a lattice. To see how this relates to the grading group A and the lattice L, we apply the functor $\mathrm{Hom}(-, \mathbb{Z})$ to the Sequence 1:

$$0 \to \mathrm{Hom}(A, \mathbb{Z}) \to \mathbb{Z}^n \to L^\vee \to \mathrm{Ext}(A, \mathbb{Z}) \to 0.$$

Let $v_1, \ldots, v_n$ denote the images in $L^\vee$ of the unit vectors in the above map, so that $L \hookrightarrow \mathbb{Z}^n$ is given by $u \mapsto (v_1 \cdot u, \cdots, v_n \cdot u)$. We say a fan Σ is **compatible** with the multigrading defined by A if all cones in Σ are generated by a subset of the $\{v_1, \ldots, v_n\}$.

Example. In our case, we have that $\{v_1, v_2, v_3, v_4\} = \{e_1, -e_1, e_3, -e_3\}$. The cone generated by $\{v_1, \ldots, v_4\}$ is all of $L^\vee \otimes R$, which means that the A-grading is positive. We will represent these as vectors in $\mathbb{R}^2$ using the isomorphism described before. Since this toric variety is projective, its fan is the inner normal fan of a polytope, and the polytope is the 2-cube $\mathrm{conv}\{0,1\}^2 \subset \mathbb{R}^2$. We say two vectors in $L^\vee \otimes \mathbb{R}$ are equivalent if they are minimized on the same face of of the cube. This defines a fan which is compatible with the grading, and it looks like the following picture.



To obtain a toric variety from this fan, given a cone σ inside $L^\vee \otimes \mathbb{R}$, where $L^\vee$ is the lattice of one parameter subgroups, we form the dual cone $\sigma^\vee$ inside $L \otimes \mathbb{R}$, the character lattice. From there, we define the **affine toric variety from σ** to be

$$U_\sigma = \mathrm{Spec}\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n].$$

Example. One cone in the fan corresponding to $\mathbb{P}^1 \times \mathbb{P}^1$ is $\sigma = \text{pos}(e_1, e_2) \subset \mathbb{Z}^2$. Then the dual cone $\sigma^\vee$ is $\text{pos}(e_1, e_2)$, and the corresponding semigroup is $\mathbb{C}[x_0, y_0]$. Then

$$U_\sigma = \text{Spec}\mathbb{C}[\sigma^\vee] = \mathbb{A}^2.$$

To obtain the entire toric variety X_Σ, we we glue the affine toric varieties according to the face structure of the fan.

3 GIT Quotients: Affine and Projective

After applying the contravariant functor $\text{Hom}(-, \mathbb{C}^*)$ to the exact sequence 1, we obtain the exact sequence (one needs to check that the last map is surjective) of multiplicative abelian groups

$$1 \to G \to (\mathbb{C}^*)^n \to \text{Hom}(L, \mathbb{C}^*) \to 1,$$

where $G = \text{Hom}(A, \mathbb{C}^*)$ is the **character group** of A. Let $z_1, \ldots, z_n$ be the coordinates on $(\mathbb{C}^*)^n$. The subgroup G of $(\mathbb{C}^*)^n$ is the common zero locus of the lattice ideal I_L, which is regarded as an ideal in the Laurent polynomial ring $\mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$. The torus acts on S by scaling the variables, and this restricts to an action of G on S.

Example. In our case, $G = \text{Hom}(A, \mathbb{C}^*) = (\mathbb{C}^*)^2$. We have that $I_L = (z_1 - z_2, z_3 - z_4)$. Then G is the subgroup of $(\mathbb{C}^*)^4$ where $z_1 = z_2$ and $z_3 = z_4$. Therefore, G acts on $\mathbb{C}^4$ by

$$(x_0, x_1, y_0, y_1) \mapsto (z_1 x_0, z_1 x_0, z_3 y_0, z_3 y_1).$$

We wish to construct a variety which is the quotient of $\mathbb{C}^n$ mod the action of G. The coordinate ring of this will consist of elements of S which are fixed under the action of G.

Lemma 1. *A polynomial $f \in S$ is fixed by G if and only if it is homogeneous of degree 0, so that $deg(f) \in L \cap \mathbb{N}^n$.*

Definition 2. The **affine GIT quotient** of $\mathbb{C}^n$ mod G is the affine toric variety $\text{Spec}(S^G)$ whose coordinate ring is S^G. This is denoted

$$\mathbb{C}/\!\!/G := \text{Spec}(S^G) = \text{Spec}(\mathbb{C}[\mathbb{N}^n \cap L]).$$

Example. In our case, $S^G = \mathbb{C}[L \cap \mathbb{N}^n] = \mathbb{C}$. This happens because the A grading is **positive**. So, the affine GIT quotient $\mathbb{C}^n/\!\!/G$ is not very interesting– it is just a point.

To remedy this problem, we will have to extend this definition so that we can obtain toric varieties which are not affine. Fix an element $a \in A$. Consider the graded components of S, given by S_{ra}, where r ranges over all nonnegative integers. We introduce an auxilliary variable γ, and define

$$S_{(a)} := \bigoplus_{r=0}^{\infty} \gamma^r S_{ra} = S_0 \oplus \gamma S_a \oplus \gamma^2 S_{2a} \oplus \cdots .$$

Example. Since our toric variety is coming from the lattice polytope given by the 2 cube, the correct choice of element $a \in A$ to obtain $\mathbb{C}^4 /\!\!/_a G = \mathbb{P}^1 \times \mathbb{P}^1$ is given in the following way. The two cube has inner normal vectors $v_1 = (1,0), v_2 = (-1,0), v_3 = (0,1), v_4 = (0,-1)$. Then, the cube is defined by equations $v_i \cdot P \geq w_i$, where $w = (0,1,0,1)$. We define a to be the coset in $\mathbb{Z}^n/L$ containing w, so we take $a = (1,1)$.

We can compute that

$$S_{(1,1)} = \mathbb{C}[\gamma x_0 y_0, \gamma x_0 y_1, \gamma x_1 y_0, \gamma x_1 y_1].$$

Definition 3. The **projective GIT quotient** of $\mathbb{C}^n$ mod G at a is the projective spectrum $\mathbb{C}^n /\!\!/_a G$ of the $\mathbb{N}$-graded S_0-algebra $S_{(a)}$:

$$\mathbb{C}^n /\!\!/_a G = \mathrm{Proj}(S_{(a)}).$$

This consists of all prime ideals in $S_{(a)}$ which are homogeneous with respect to γ and do not contain the **irrelevant ideal**

$$S_{(a)}^{+} = \bigoplus_{r=1}^{\infty} \gamma^r S_{ra}.$$

This has a cover by open affines which we call $U(x^u\gamma^r)$, one for each generator of $S_{(a)}$ over S_0. This affine open subset consist of all points in $\mathbb{C}^n /\!\!/_a G$ for which the coordinate $x^u\gamma^r$ is nonzero. More precisely, it is the spectrum of the $\mathbb{C}$-algebra consisting of elements of γ-degree 0 in the localization of $S_{(a)}$ by inverting $x^u\gamma^r$.

Example. Our toric variety $\mathbb{C}^4 /\!\!/_{(1,1)} G$ is covered by four open affine varieties, one for each generator of $S_{(1,1)}$ over $S_0 = \mathbb{C}$. For example,

$$U(\gamma x_0 y_0) = \mathrm{Spec}(\mathbb{C}[y_1/y_0, x_1/x_0])$$

and the spectrum of this is $\mathbb{A}^2$. The four generators each correspond to one copy of $\mathbb{A}^2$.

4 Toric Varieties as Quotients

Recall from earlier the fan Σ inside $L^\vee$. We define the **irrelevant ideal of Σ** to be the ideal

$$B_\Sigma = (x_{j_1} \cdots x_{j_s} \mid \{v_1, \ldots, v_n\} \backslash \{v_{j_1} \cdots v_{j_s}\} \text{ spans a cone of } \Sigma).$$

The variety $V(B_\Sigma)$ consists of coordinate subspaces of $\mathbb{C}^n$.

Example. The irrelevant ideal (and its primary decomposition) in this case is

$$B_\Sigma = (x_0y_0, x_0y_1, x_1y_0, x_1y_1) = (x_0, x_1) \cap (y_0, y_1)$$

Let $U_\sigma = \mathbb{C}^n \backslash V(x^{\overline{\sigma}}) = \mathrm{Spec}(S[x^{-\overline{\sigma}}])$. By taking the degree 0 components, we obtain affine GIT quotients $X_\sigma = U_\sigma /\!\!/ G = \mathrm{Spec}(S[x^{-\overline{\sigma}}]_0)$.

Proposition 1. *The affine toric variety X_σ is equal to the spectrum semigroup $\sigma^\vee \cap L$ where $\sigma^\vee$ is the cone in $L \otimes \mathbb{R}$ dual to σ.*

Definition 4. Let B be an irrelevant ideal for some compatible fan Σ. The the **toric variety X_Σ with homogeneous coordinate ring** S and irrelevant ideal B, is the variety covered by the affine toric varieties X_σ for cones $\sigma \in \Sigma$.

Now we wish to study what "quotentlike" properties X_Σ has.

Definition 5. A variety X is called a **categorical quotient** of a variety U mod the action of a group G if there is a G equivariant morphism $U \to X$ in which X carries the trivial G action, with the property that any G-equivariant morphism $U \to Y$ with trivial G action factors uniquely $U \to X \to Y$.

Theorem 1. *The toric variety X_Σ is the categorical quotient of $\mathbb{C}^n \backslash V(B_\Sigma)$ by G.*

In the case that we are dealing with a projective toric variety, Σ is the inner normal fan of a polytope P. Assume P has full dimension $n - d = rank(L)$. Then P only intersects some faces of $\mathbb{R}^n_{\geq 0}$. We say P **misses** the other faces.

Theorem 2. *The normal fan Σ of P is compatible, and its irrelevant variety $V(B_\Sigma)$ corresponds to the simplicial complex of faces missed by P. The projective GIT quotient $\mathbb{C}^n /\!\!/_a G$ is the toric variety X_Σ, where a is as defined before.*

Example. In our case, the polyhedron P is the 2-cube

$$\mathrm{conv}((1,0,1,0),(0,1,1,0),(1,0,0,1),(0,1,0,1)).$$

The dimension is 2 which is equal to the rank of L. We see that P intersects the faces $x_0 = 0$, $x_1 = 0$, $y_0 = 0$, $y_1 = 0$, $x_0, y_0 = 0$, $x_1, y_0 = 0$, $x_0, y_1 = 0$, $x_1, y_1 = 0$, and does not intersect the faces $x_0, x_1 = 0$ or $y_1, y_0 = 0$ or any higher dimensional faces. Then, this corresponds to the variety $V(x_0y_0, x_1y_0, x_0y_1, x_1y_1)$, which was the irrelevant ideal.

The projective GIT quotient $\mathbb{C}^n /\!\!/_{(1,1)} G$ is the toric variety $X_\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$.

Library of Symbols

L	The character lattice.
$L^\vee$	The lattice of one parameter subgroups.
Σ	a fan inside $L^\vee$
σ	cones inside the fan Σ
U_σ	The affine toric variety corresponding to σ
S	$\mathbb{C}[x_1, \ldots, x_n]$ the polynomial ring with grading by A
A	the grading group
G	the character group of A, $\mathrm{Hom}(A, \mathbb{C}^*)$
S^G	Elements of S which are fixed by G
$\mathbb{C}^n /\!\!/ G$	The affine GIT quotient of $\mathbb{C}^n$ by G
a	An element of A.
$\mathbb{C}^n /\!\!/_a G$	The projective GIT quotient of $\mathbb{C}^n$ by G with respect to a.
$v_1, \ldots, v_n$	Images in $L^\vee$ of the unit vectors of $\mathbb{Z}^n$. They generate the cones in the fan Σ.
$\sigma^\vee$	Dual cone of σ
X_Σ	The toric variety defined by the fan Σ.
I_L	Lattice ideal of the lattice L
P	A polytope (to which Σ is sometimes the inner normal fan)