Tropical Geometry of Curves

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1 Introduction

Tropical geometry is a new subject which creates a bridge between the two islands of algebraic geometry and combinatorics. It has many fascinating connections to other areas as well. The aim of this lecture is to introduce tropical geometry and to provide a glimpse of a research area in tropical geometry, namely, computing abstract tropicalizations of curves.

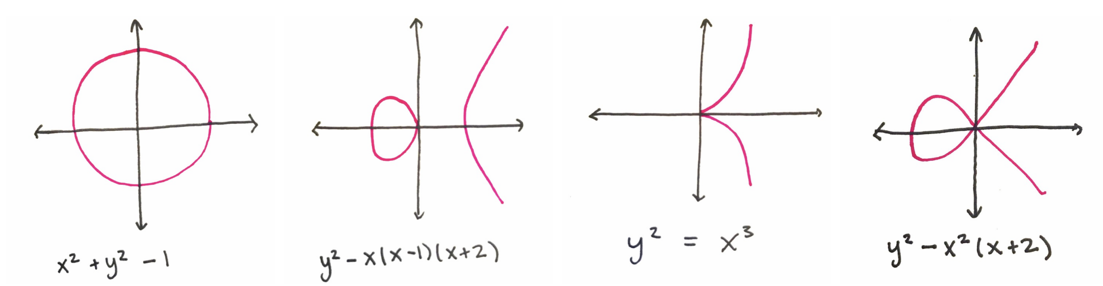
2 Background

To start, let K be a field.

Definition 2.1. Let $f \in K[x, y]$ be a polynomial. A **plane algebraic curve** over K is a set of the form

$$\{(x, y) \in K^2 \mid f(x, y) = 0\}.$$

Example 2.2. I now give four examples of plane curves in $\mathbb{R}^2$.



The first two curves are **smooth** and the last two are **singular**. The left one is called a **cusp** and the right one is called a **node**.

We will work over a special field called the Puiseux series for tropical geometry. Doing this allows you to encode a family of curves as one curve. This field is also nice because it is algebraically closed, which is important for counting in algebraic geometry.

Definition 2.3. The **Puiseux series** $\mathbb{C}\{\{t\}\}$ is:

$$\{c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \cdots \mid c_1 \neq 0, \quad a_i \text{ an increasing sequence of rational numbers which have a common denominator}\}.$$

The Puiseux series are equipped with a special function $\nu : K \to \mathbb{R} \cup \{\infty\}$, called the **valuation**. The valuation is given by taking $\nu(c) = a_1$ and $\nu(0) = \infty$.

Tropicalization is a process we apply to varieties over a field with a valuation. The Puiseux series are not the only field with a valuation, but to simplify matters we only use this example in this talk.

3 Embedded tropicalization of plane curves

Definition 3.1. Given a polynomial

$$f(x, y) = \sum_{(a,b)\in\mathbb{Z}^2} c_{a,b} x^a y^b,$$

we define its **tropicalization** to be the real valued function $\mathrm{trop}(f) : \mathbb{R}^2 \to \mathbb{R}$ that is obtained by replacing each c_u by its valuation and preforming all additions and multiplications in the **tropical semiring** $(\mathbb{R}, \oplus, \otimes)$:

$$\mathrm{trop}(f)(x, y) = \min_{(a,b)\in\mathbb{Z}^2} \left(\mathrm{val}(c_{(a,b)}) + ax + by\right).$$

Example 3.2. Consider the polynomial $f = t + t^2 x + y^3 \in \mathbb{C}\{\{t\}\}[x, y]$. Then

$$\begin{aligned}\mathrm{trop}(f)(x, y) &= \min(val(t), val(t^2) + (1, 0) \cdot (x, y), val(1) + (0, 3) \cdot (x, y)) \\ &= \min(1, 2 + x, 3y)\end{aligned}$$

We saw earlier that f defines a plane algebraic curve over the Puiseux series $\mathbb{C}\{\{t\}\}$. Now we will see how to make a tropical curve associated to $\text{trop}(f)$.

Definition 3.3. The **tropical curve** associated to $\text{trop}(f)$ is the set

$$\{(x, y) \in \mathbb{R}^2 \mid \text{the minimum in } \text{trop}(f)(x, y) \text{ is achieved at least twice}\}$$

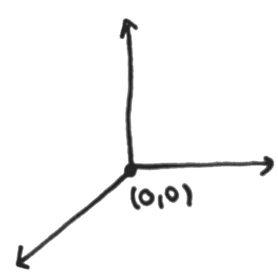
Example 3.4. Here we compute the **tropical line**, which is the classic first example. Let $f = x + y + 1$ in the field $\mathbb{C}\{\{t\}\}$. Then,

$$\begin{aligned} \text{trop}(f)(x, y) &= \min\left(0 + (1, 0) \cdot (x, y), 0 + (0, 1) \cdot (x, y), 0\right), \\ &= \min(x, y, 0). \end{aligned}$$

So, where is this minimum achieved twice? We can break this down in to 3 cases.

1. x and 0 are the winners: This happens when $x = 0$ and $y \geq 0$. So, this is the ray $\text{pos}(e_2)$.
2. y and 0 are the winners: This happens when $y = 0$ and $x \geq 0$. So, this is the ray $\text{pos}(e_1)$.
3. x and y are the winners: This adds to our tropical variety the ray $\text{pos}(-1, -1)$.

So, the tropical variety is as pictured below.



Problem 3.5. If we do embedded tropicalizations for a curve defined by a polynomial $f(x, y)$ and again for $f(ax + by + c, dx + ey + f)$ after a simple change of coordinates, we can get wildly different answers. Morally, changing coordinates should not change any intrinsic properties of the curve.

Question 3.6. Is there a way to associate a tropical object to a curve that is invariant under coordinate transformation?

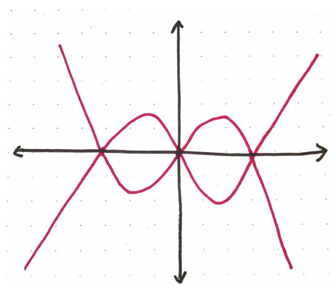
4 Abstract Tropicalization

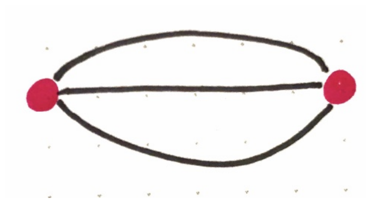
Now, our goal is to define an object called the **abstract tropicalization** which can be associated to every curve, and does not depend on the embedding.

Let X be a smooth plane curve over $\mathbb{C}\{\{t\}\}$ defined by the polynomial f. The coefficients of f depend on the parameter t. Informally, we can think of the parameter t as "going to zero," and when $t = 0$, we will see some special and possibly singular behavior. For general values of t, the curve will be smooth, but it could limit to something singular. If the singularities are only nodes, then we can define the **dual graph**.

Definition 4.1. The **abstract tropicalization** of X has vertices corresponding to the irreducible components of X, and edges corresponding to nodes.

Example 4.2. Consider the curve defined by $f(x) = y^2 - x^2(x-1)^2(x+1)^2$. It is shown on the left, and its dual graph is shown on the right.





So, why is this hard? Depending upon the equations that the curve arrives to you with, the singularities when $t = 0$ could be worse than nodes. By the **semistable reduction theorem**, we are always guaranteed in the abstract that we can correct this. However, this process is not algorithmic, and this proves to be the main difficulty in finding the abstract tropicalization of a curve in explicit examples.

The problem of computing the abstract tropicalization has been studied in several classes of curves and by multiple approaches.

Theorem 4.3 (Bolognese-B-Chua, B-Helminck)**.** *There is an algorithm for computing the abstract tropicalization of hyperelliptic curves ($y^2 = g(x)$) and supelliptic curves ($y^n = g(x)$).*

Example 4.4. We use as a running example finding the abstract tropicalization of the curve defined by the polynomial

$$f(x, y) = y^2 - x(x-t)(x-1)(x-1-t)(x+1)(x+1-t)$$

When $t = 0$, this gives the previous example. So, we already know the abstract tropicalization, but let's compute it in a different way.

The map $(x, y) \mapsto x$ from the curve $X \to \mathbb{C}\{\{t\}\}$ is a two-to-one map. This means that almost every point in $\mathbb{C}\{\{t\}\}$ has two preimages, except some special points. These are the six roots of g:

$$\{0, t, 1, 1+t, -1, -1+t\}.$$

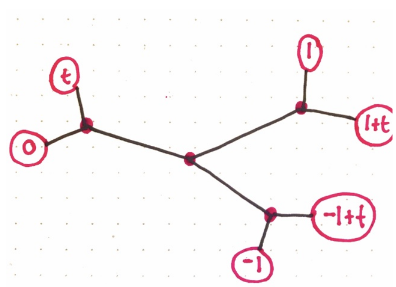
We think of these points as "marking" $\mathbb{C}\{\{t\}\}$.

The entire situation of our two-to-one map $X \to \mathbb{C}\{\{t\}\}$ tropicalizes, meaning that there will be two grpahs $\Gamma \to T$ that form a two-to-one cover. The graph T is the tropicalization of $\mathbb{C}\{\{t\}\}$ with its marked points.

Let m_{ij} be the difference between the ith and the jth root, and $d_{ij} = 2 - 2\nu(m_{ij})$.

	12	13	14	15	16	23	24	25	26	34	35	36	45	46	56
m	$-t$	-1	$-1-t$	1	$1-t$	$-1+t$	-1	$1+t$	1	$-t$	2	$2-t$	$2+t$	2	$-t$
d	0	2	2	2	2	2	2	2	2	0	2	2	2	2	0

The number d_{ij} is the distance between leaf i and leaf j in the tree T. These distances uniquely specify the tree T, and one can use the Neighbor Joining Algorithm to reconstruct the tree T from these distances. Therefore, the tree is as displayed below.



The map from the graph we are looking for to this tree must satisfy some special properties. It has to be a degree 2 harmonic morphism. In degree 2, there is only one option, which we saw before.