Tensors and their Eigenvectors

Madeline Brandt

June 9 2019

Tensors are higher dimensional analogs of matrices. We will see that one way to view a symmetric tensor is as a homogeneous polynomial. Basic attributes of matrices, like eigenvectors, can be defined for tensors. This talk is split in 2 sections– symmetric and non symmetric tensors. For each, we review some familiar aspects of matrices in preparation for the analagous concept for tensors.

Definition 1. A **tensor** is a d-dimensional array $T = (t_{i_1,\ldots,i_d})$. The entries are elements of the ground field K. The set of all tensors of format $n_1 \times \cdots \times n_d$ form a vector space of dimension $n_1 \cdots n_d$ over K.

1 Symmetric Tensors, Homogeneous Polynomials, Eigenvectors

1.1 Square Symmetric Matrices (d = 2)

Let K be a field. Recall that symmetric matrices correspond to quadratic forms.

Example 2. Let $Q = 2x^2 + 7y^2 + 23z^2 + 6xy + 10xz + 22yz$. This is represented as a symmetric 3×3-matrix as follows:

$$Q = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 2 & 3 & 5 \\ 3 & 7 & 11 \\ 5 & 11 & 23 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The **gradient** of a quadratic form is the vector of its partial derivatives. So, it is a vector of linear forms, giving a map $K^n \to K^n$.

Example 3. For the quadratic form we have from before, this is given by

$$\nabla Q = \begin{pmatrix} \partial Q/\partial x \\ \partial Q/\partial y \\ \partial Q/\partial z \end{pmatrix} = 2 \cdot \begin{pmatrix} 2 & 3 & 5 \\ 3 & 7 & 11 \\ 5 & 11 & 23 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Then, a vector $v \in K^n$ is an **eigenvector** of Q if v is mapped to a scalar multiple of v: $\nabla Q v = \lambda v$, $\lambda \in K$. Replacing K^n by projective space $\mathbb{P}^{n-1}$, we obtain a rational self-map of projective space:

$$\nabla Q : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}.$$

If Q is rank-deficient then the linear map has a kernel. These are places where the gradient ∇Q vanishes. These are called **base points** of the map. If Q has full rank then ∇Q is a regular map $\mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ so it is defined on all of $\mathbb{P}^{n-1}$

Question 4. If you are interested in trying an example, do Question 1.

Remark 5. The eigenvectors of Q are the fixed points ($\lambda \neq 0$) and base points ($\lambda = 0$) of the gradient map ∇Q.

1.2 Symmetric Tensors

An $n \times \cdots \times n$ tensor $T = (t_{i_1,\ldots,i_d})$ is called **symmetric** if it is unchanged after permuting the indices. Symmetric tensors correspond to homogeneous polynomials of degree d in n variables:

$$T = \sum_{i_1,\ldots,i_d=1}^{n} t_{i_1,\ldots,i_d} \cdot x_{i_1} \cdots x_{i_d}.$$

Remark 6. For the rest of this section, it is more convenient to think of a tensor as a polynomial, NOT as an array.

As with matrices, the gradient of T defines a map $\nabla T : K^n \to K^n$ (T is a homogeneous polynomial in n variables of degree d).

Definition 7. A vector $v \in K^n$ is an **eigenvector** of T if $(\nabla T)(v) = \lambda v$ for $\lambda \in K$.

Question 8. If you would like to compute an example, do Question 4.

If we again think of this map instead as a map on $\mathbb{P}^{n-1}$, then the gradient map is a rational map from projective space to itself:

$$\nabla T : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}.$$

The eigenvectors of T are **fixed points** ($\lambda \neq 0$) and **base points** ($\lambda = 0$) of ∇T.

Theorem 9 (Cartwright-Sturmfels)**.** *If K is algebraically closed, then the number of eigenvectors of a general d-dimensional $n \times \cdots \times n$ symmetric tensor T is*

$$\frac{(d-1)^n - 1}{d-2} = \sum_{i=0}^{n-1} (d-1)^i.$$

Proof. The proof is Question 5. □

Example 10. (n = d = 3) Consider the Fermat Cubic $T = x^3 + y^3 + z^3$. Its gradient map is the regular map that squares each coordinate:

$$\nabla T : \mathbb{P}^2 \to \mathbb{P}^2, \ (x : y : z) \mapsto (x^2 : y^2 : z^2).$$

This has $7 = 1 + 2 + 2^2$ fixed points (all combinations of 1,0 minus all 0's):

$$(1:0:0), (0:1:0), (0:0:1), (1:1:0), (1:0:1), (0:1:1), (1:1:1).$$

Therefore, T has 7 eigenvectors, as the theorem predicts.

2 Rectangular Tensors, Multilinear Forms, Singular Vectors

2.1 Rectangular matrices (d = 2)

For a rectangular matrix, one instead considers **singular vectors**. The number of singular vectors is equal to the smaller of the two matrix dimensions. Each rectangular matrix represents a bilinear form.

Example 11. Consider the following bilinear form.

$$B = 2ux + 3uy + 5uz + 3vx + 7vy + 11vz = \begin{pmatrix} u & v \end{pmatrix} \begin{pmatrix} 2 & 3 & 5 \\ 3 & 7 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Its gradient defines an endomorphism on the direct sum of the row space and the column space. We get a map $\nabla B : K^2 \oplus K^3 \to K^2 \oplus K^3$ sending the pair

$$((u,v),(x,y,z)) \mapsto \left(\left(\frac{\partial B}{\partial u},\frac{\partial B}{\partial v}\right),\left(\frac{\partial B}{\partial x},\frac{\partial B}{\partial y},\frac{\partial B}{\partial z}\right)\right)$$
$$= ((2x+3y+5z, 3x+7y+11z),(2u+3v,3u+7v,5u+11v))$$

Let B be an $m \times n$ matrix over K. Consider the equations

$$Bx = \lambda y, \qquad B^t y = \lambda x$$

for $\lambda \in K$, $x \in K^n$, $y \in K^m$. Given a solution to these equations, we see that x is an eigenvector of B^tB, y is an eigenvector of BB^t, and λ^2 is a common eigenvalue. We call x, y the **right and left singular vector**.

Remark 12. The singular pairs (x,y) of a rectangular matrix B are fixed points of the gradient map ∇B of the associated bilinear form. This is now a self-map on the product of projective spaces:

$$\nabla B : \mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$$

Question 13. For those interested in computing an example, do Question 2.

2.2 Rectangular Tensors

Consider now a d-dimensional tensor T in $K^{n_1 \times \cdots \times n_d}$. It corresponds to a multilinear form.

Definition 14. The **singular vector tuples** of T are the fixed points of the gradient map

$$\nabla T : \mathbb{P}^{n_1-2} \times \cdots \times \mathbb{P}^{n_d-2} \to \mathbb{P}^{n_1-2} \times \cdots \times \mathbb{P}^{n_d-2}.$$

Example 15. The trilinear form $T = x_1y_1z_1 + x_2y_2z_2$ is interpreted as a $2 \times 2 \times 2$ tensor. The gradient ∇T of this trilinear form is the rational map

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\ ((x_1 : x_2),(y_1 : y_2),(z_1 : z_2)) & \mapsto & ((y_1z_1 : y_2z_2),(x_1z_1 : x_2z_2),(x_1y_1 : x_2y_2)). \end{array}$$

This map has six fixed points, for example $((1:0),(1:0),(1:0))$, and others. These are the singular vector triples of the tensor T.

The expected number of singular vector triples is predicted by the following theorem.

Theorem 16 (Friedland and Ottaviani)**.** *For a general* $n_1 \times \cdots \times n_d$*-tensor* T *over an algebraically closed field* K*, the number of singular vector tuples is the coefficient of the monomial* $z_1^{n_1-1} \cdots z_d^{n_d-1}$ *in the polynomial*

$$\prod_{i=1}^{d} \frac{(\hat{z}_i)^{n_i} - z_i^{n_i}}{\hat{z}_i - z_i},$$

where $\hat{z}_i = z_1 + \cdots + z_{i-1} + z_{i+1} + \cdots + z_d$.

Example 17. (Question 3) Consider the $3 \times 3 \times 2 \times 2$ tensor defined by the multilinear form $T = x_1y_1z_1w_1 + x_2y_2z_2w_2$.

Computing the polynomial in the above theorem and examining the coefficient of the monomial $x_1^2y_1^2z_1w_1$, we expect that there are 98 singular vector tuples for T.

We will now determine all singular vectors of T. The gradient map sends

$$((x_1 : x_2 : x_3), (y_1 : y_2 : y_3), (z_1 : z_2), (w_1 : w_2)) \quad \mapsto$$

$$((y_1z_1w_1 : y_2z_2w_2 : 0), (x_1z_1w_1 : x_2z_2w_2 : 0), (x_1y_1w_1 : x_2y_2w_2), (x_1y_1z_1 : x_2y_2z_2)).$$

What are the fixed points of this map? First, we observe that $x_3, y_3 = 0$.

If $x_1 = 0$: Then $y_1 = z_1 = w_1 = 0$, so the only solution is $((0, 1, 0), (0, 1, 0), (0, 1), (0, 1))$.

If $x_1 \neq 0$: Then $y_1z_1w_1 \neq 0$. So, we may set $x_1 = y_1 = z_1 = w_1 = 1$. Then we obtain:

$$((1 : x_2 : 0), (1 : y_2 : 0), (1 : z_2), (1 : w_2))$$

$$= ((1 : y_2z_2w_2 : 0), (1 : x_2z_2w_2 : 0), (1 : x_2y_2w_2), (1 : x_2y_2z_2))$$

Macaulay2 (degree + primary decomposition) reveals that there are 17 solutions, and 9 of them are real. So in total, we have 18 singular vector tuples.