Symmetric powers of algebraic and tropical curves

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Tropical geometry gives a way to connect algebraic geometry and combinatorics or polyhedral geometry. Today I will give an example of this by studying the collection of divisors on a curve or graph.

Throughout, let K be a non-Archimedean field with valuation ring R whose residue field k is algebraically closed and contained in K. Let X be a smooth projective curve over K of genus $g \geq 1$ and let $d \geq 0$. An **effective degree d divisor** on a X is a finite formal sum of the form $\sum n_i v_i$ where the n_i are positive integers summing to d, and $v_i \in X$.

The **d-th symmetric power** X_d of X is defined to be the quotient

$$X_d = X^d / S_d$$

of the d-fold product $X^d = X \times \cdots \times X$ by the action of the symmetric group S_d that permutes the entries. The symmetric power X_d is again a smooth and projective algebraic variety and functions as the **moduli space of effective divisors of degree d on X**.

A **tropical curve** $\Gamma = (G, V, l, w)$ is a metric graph with some weights. One way to think about this is through models. A **model** of a metric graph is a graph $G = (V, E)$ and a "length function" l on the edges so that the metric graph is obtained by gluing together intervals of the correct length according to the instructions given by the graph. To make it a tropical curve, we also add weights to the vertices (the role of these weights will be explained later).

Then, a **divisor** on a tropical curve is again a finite formal sum of the form $\sum n_i v_i$, and the d-th symmetric power is the quotient

$$X_d = X^d / S_d.$$

What follows is the main theorem that I will explain in this talk. The remainder of the talk will be dedicated to understanding the statement in more detail, and a sketch of the proof will be given at the end.

Theorem 1 (M-Ulirsch)**.** *The non-Archimedian skeleton of the effective degree d-divisors on a curve is the effective degree d divisors on the skeleton of the curve.*

Are there any questions up to this point?

1 Skeletons of curves

First, I will say how to tropicalize an algebraic curve X and get a tropical curve Γ.

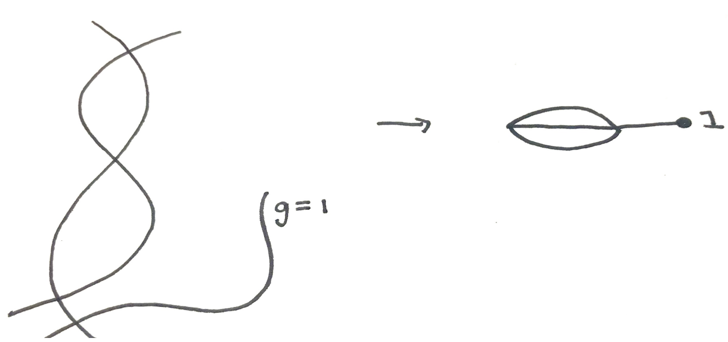
Madeline: [Depending upon what is said in the introductory talks, say more or less about the "intuitive version" of this]

We say a **model** $\mathcal{X}$ for a curve X is a flat and finite type scheme over R ($\mathrm{Spec} R = \{0, m\}$) whose **generic fiber** (fiber over (0)) is isomorphic to X. We call this model **strictly semistable** if the **special fiber** (fiber over the maximal ideal) satisfies:

1. It is reduced, connected, and only has nodal singularites;
2. every rational component meets the rest of the curve in at least 2 singular points (and no self-intersection).

Definition 1. The **dual graph** G of $\mathcal{X}_k$ has vertices corresponding to the irreducible components of $\mathcal{X}_k$, and edges corresponding to nodes.

Here is an example of a schematic of the special fiber of a curve G and on the right, its dual graph.



To define the **tropicalization** Γ of the curve $\mathcal{X}$, we add a bit of extra data.

1. **Vertex Weights**: we add weights to the vertices by assigning to each vertex the **genus** of the corresponding component.

2. **Edge Lengths**: We add edge lengths in the following way. Given an edge corresponding to a node q between two components X_i and X_j, the completion of the local ring $\mathcal{O}_{\mathcal{X},q}$ is isomorphic to $R[[x,y]]/(xy - f)$ where $\nu(f) > 0$. Then, we define the length of the edge e_{ij} to be $\nu(f)$.

2 Divisors on a tropical curve. What is Γ_d?

Motivating Principle: When finding the skeleton of the curve, we just observed that there was a correspondence between:

$$\text{strata in the special fiber} \quad \leftrightarrow \quad \text{cells in a polyhedral complex}$$

The same story will hold for X_d. We make a **nice model** (which I will not describe) so that the strata of the special fiber of this model are dual to some polyhedral cells (which I will now describe).

Recall from earlier that Γ_d is the set of effective degree d divisors on Γ– these are just formal linear combinations of points on Γ with positive coefficients which sum to d.

We would like to think of it as a **colored polysimplicial complex**, to add more combinatorial structure that will agree with the skeleton of X_d. A colored polysimplicial complex is like a simplicial complex, but now our basic building blocks include both simplices and products of simplices (like squares, toblerones, etc). A polysimplex formed as a product of k simplices is colored by a vector of positive real numbers of length k which we think of as recording the volume of each simplex. For example, a toblerone bar of chocolate is $(\Delta_2 \times \Delta_1, (1,9))$ and an $Apieceofpaperis (\Delta_1 \times \Delta_1, (210, 297))$ (mm).

We now describe the colored polysimplicial complex structure on Γ_d. Given G, the dual graph of our fixed semistable model $\mathcal{X}$, and a degree d, consider the poset of **stable pairs** (G', D) over G, where G' subdivides G and $D(\nu) > 0$ for all exceptional vertices $\nu \in G$. Associate to (G', D) the polysimplex

$$(\Delta_{k_1} \times \cdots \times \Delta_{k_l}, (l(e_1), \ldots, l(e_l)))$$

where k_i is the number of vertices living above the edge e_i. This polysimplex parametrizes all divisors on the metric graph Γ with combinatorial type (G', D).

Example 2. To the G' in this picture we associate a rectangle of size $l_1 \times l_2$. This rectangle gives all of the divisors on Γ of type (G', D).

We glue these cells according to the poset of stable pairs to obtain the colored polysimplicial complex structure on Γ_d.

3 What is the tropicalization of X_d?

Now, in order to tropicalize X_d, we must also find a nice model. This is now called a **polystable model**, and we will denote it $\mathcal{X}_d$. I will describe what the strata of the special fiber look like.

In the generic fiber of $\mathcal{X}_d$ we have X_d. On the other hand, points in the special fiber of $\mathcal{X}_d$ are given by a pair $(\mathcal{X}', \mathcal{D})$ satisfying:

1. the generic fiber of $\mathcal{D}$ is D,
2. the support of $\mathcal{D}_0$ in the special fiber does not meet the nodes of $\mathcal{X}_0'$
3. the support of $\mathcal{D}_0$ meets every exceptional component of $\mathcal{X}_0'$ over $\mathcal{X}_0$.

To this we may associate the pair $(G', \text{mdeg} D)$, where

$$\text{mdeg}(\mathcal{D}) = \sum_{v \in V(G')} \deg(\mathcal{D}|_{X_v'}) \cdot v.$$

Then G' is a subdivision of G and $(G', \text{mdeg} D)$ if a stable pair over G.

The strata of $(\mathcal{X}_d)_0$ are exactly the loci where the dual pairs are constant. To tropicalize X_d, we form a polysimplicial complex which encodes the combinatorial data of how the strata intersect.

In the end, **We have an order preserving, 1-1 correspondence between the strata and the stable pairs**.

(The model we take for X_d is $\operatorname{Spec} R \times_{\overline{M}_g} \overline{\operatorname{Div}_{g,d}}$ where the map $\operatorname{Spec} R \to \overline{M_g}^{ss} \to \overline{M}_g$ identifies a strictly semistable model for X and then stabilizes, and $\overline{\operatorname{Div}_{g,d}}$ is the moduli space whose fiber over a family of stable curves $X \to \operatorname{Spec} R$ is the set of pairs (X', D) consisting of a semistable model X' of X and a divisor D on X' such that the support of D does not meet the nodes of X' and the support of D does meet every exceptional component of X'.)