Symmetric powers of algebraic and tropical curves

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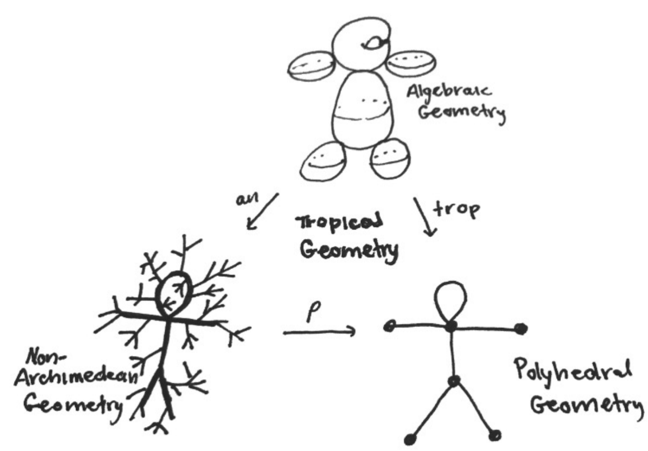
Abstract

We show that the non-Archimedean skeleton of the d-th symmetric power of a smooth projective algebraic curve X is naturally isomorphic to the d-th symmetric power of the tropical curve that arises as the non-Archimedean skeleton of X.

Tropical geometry gives us a way to unite the worlds of algebraic geometry, non-archimedean geometry, and polyhedral geometry. Many results in this area follow the following format:



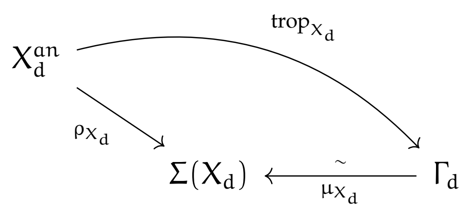
In which the map "trop" associates to an algebraic object the "intuitively defined" tropical version, the map "an" gives the Berkovich analytification, and the map ρ gives the retraction onto the skeleton. Then, one must prove that the skeleton equals the intuitively defined polyhedral object.

Throughout, let K be a non-Archimedean field with valuation ring R whose residue field k is algebraically closed and contained in K. Let X be a smooth projective curve over K of genus $g \geq 1$ and let $d \geq 0$. Let $\mathcal{X}$ be a fixed strictly semistable model of X over R. Denote by Γ the dual tropical curve of X.

Theorem 1. *There is a natural isomorphism*

$$\mu_{X_d} : \Gamma_d \xrightarrow{\sim} \Sigma(X_d)$$

of colored polysimplicial complexes that makes the diagram



commute.

1 What is Γ_d?

We say a **model** $\mathcal{X}$ for a curve X is a flat and finite type scheme over R whose generic fiber is isomorphic to X. We call this model **strictly semistable** if the special fiber is a strictly semistable curve over k, meaning:

1. It is reduced, connected, and only has nodal singularites;

2. every rational component meets the rest of the curve in at least 2 singular points (and no self-intersection).

The **tropical curve** G **/ dual graph** Γ of $\mathcal{X}_k$ has vertices corresponding to the irreducible components of $\mathcal{X}_k$, and edges corresponding to nodes. It also has:

1. **Vertex Weights**: assign to each vertex the (geometric) **genus** of the corresponding component.

2. **Edge Lengths**: Given an edge corresponding to a node q between two components X_i and X_j, the completion of the local ring $\mathcal{O}_{\mathcal{X},q}$ is isomorphic to $R[[x, y]]/(xy - f)$ where $\nu(f) > 0$. Then, we define the length of the edge e_{ij} to be $\nu(f)$.

An **effective degree d divisor** on a (tropical or classical) curve X is a finite formal sum of the form $\sum n_i v_i$ where the n_i are positive integers summing to d, and $v_i \in X$.

The **d-th symmetric power** X_d of a (tropical or classical) curve X is defined to be the quotient

$$X_d = X^d / S_d$$

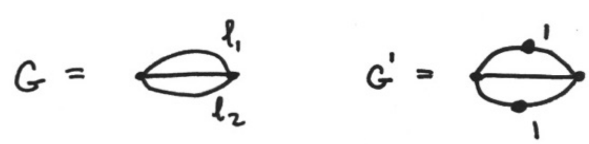
of the d-fold product $X^d = X \times \cdots \times X$ by the action of the symmetric group S_d that permutes the entries. The symmetric power X_d is again a smooth and projective algebraic variety and functions as the moduli space of effective divisors of degree d on X.

We have now defined Γ_d as a set, but we would like to think of it as a **colored polysimplicial complex**. This is like a simplicial complex, but now our basic building blocks include both simplices and products of simplices (like squares, toblerones, etc). A polysimplex formed as a product of k simplices is colored by a vector of positive real numbers of length k which we think of as recording the volume of each simplex. For example, a toblerone bar of chocolate is $(\Delta_2 \times \Delta_1, (1, 9))$.

We now describe the colored polysimplicial complex structure on Γ_d. Given G, the dual graph of our fixed semistable model $\mathcal{X}$, and a degree d, consider the poset of **stable pairs** (G', D) over G, where G' subdivides G and $D(v) > 0$ for all exceptional vertices $v \in G$. Associate to (G', D) the polysimplex

$$(\Delta_{k_1} \times \cdots \times \Delta_{k_l}, (l(e_1), \ldots, l(e_l)))$$

where k_i is the number of vertices living above the edge e_i. This polysimplex parametrizes all divisors on the metric graph Γ with combinatorial type (G', D).



Example 1. To the G' in this picture we associate a rectangle of size $l_1 \times l_2$. This rectangle gives all of the divisors on Γ of type (G', D).

We glue these cells according to the poset of stable pairs to obtain the colored polysimplicial complex structure on Γ_d.

2 What is X_d^{an}?

For each $y \in X_d$ let $\kappa(y)$ denote the residue field. As a set, X_d^{an} is

$$\{(y, |\cdot|) \mid y \in X_d \text{ and } |\cdot| \text{ is a norm on } \kappa(y) \text{ extending } \exp(-\mathrm{val}_k(\cdot)) \text{ on } K\}.$$

It has a topology given by the coarsest topology such that for any open $U \subset X_d$ and any regular function $g \in \mathcal{O}_{X_d}(U)$, the map sending $(y, |\cdot|) \mapsto |g(y)|$ is continuous.

3 What is $\Sigma(X_d)$?

Berkovich spaces are very hairy but they have the following nice property. They admit retractions onto what is called the **skeleton**, which we denote $\Sigma(X_d)$. This is given by the retraction map $\rho : X_d^{an} \to \Sigma(X_d)$. For brevity I will not write the formula here, but it is writen out explicitly in the paper.

4 The trop_{X_d} map

A point $(y, |\cdot|) \in X_d^{an}$ can be represented by a a map Spec $\kappa(y) \to X_d$. This gives us an effective divisor D on $X_{\kappa(y)}$. Then, there is a unique semistable model $\mathcal{X}' \to \mathcal{X}$ and divisor $\mathcal{D}$ in X' such that

1. the generic fiber of $\mathcal{D}$ is D,
2. the support of $\mathcal{D}_0$ in the special fiber does not meet the nodes of $\mathcal{X}_0'$
3. the support of $\mathcal{D}_0$ meets every exceptional component of $\mathcal{X}_0'$ over $\mathcal{X}_0$.

So, to the point $y \in X_d^{an}$ we associate the divisor mdeg(D) on the dual graph Γ of $\mathcal{X}$ given by

$$\mathrm{mdeg}(\mathcal{D}) = \sum_{v \in V(G')} \deg(\mathcal{D}|_{X_v'}) \cdot v$$

where G' is the dual graph of $\mathcal{X}'$. This gives the map trop_{X_d}.

5 A few words about the proof

Now we have seen each of the main objects in the statement of the theorem as well as the maps between them. A theorem of Berkovich plays a main role in the proof. It says that the skeleton is equal to a dual complex to the strata of the special fiber of a **polystable model of** X_d.

The model we take for X_d is $\operatorname{Spec} R \times_{\overline{M}_g} \overline{\mathrm{Div}_{g,d}}$ Where the map $\operatorname{Spec} R \to \overline{M_g}^{ss} \to \overline{M}_g$ identifies a strictly semistable model for X and then stabilizes, and $\overline{\mathrm{Div}_{g,d}}$ is the moduli space whose fiber over a family of stable curves $X \to \operatorname{Spec} R$ is the set of pairs (X', D) consisting of a semistable model X' of X and a divisor D on X' such that the support of D does not meet the nodes of X' and the support of D does meet every exceptional component of X'.

The main step in the proof is to show that the polysimplicial structure on Γ_d described before is the same as the one given on $\Sigma(X_d)$ given by the Berkovich theorem. To that end, we now describe the polysimplicial structure we associate to the special fiber of this model.

The special fiber $(\mathcal{X}_d)_0$ can be written as a disjoint union of strata. The strata carry a partial order:

$$E \prec E' \text{ when } E' \subset \overline{E}.$$

Around each stratum we have small charts which look like $\operatorname{Spec} A_1 \otimes_R \cdots \otimes_R A_r$ over R where A_i is of the form $R[t_1, \ldots, t_{n_i}]/(t_1 \cdots t_{k_i} - a_i)$ for $a_i \in R$.

So, we associate a stratum

$$E \mapsto (\Delta_{k_1} \times \cdots \times \Delta_{k_r}, (a_1, \ldots, a_r)).$$

This defines a complex $\Delta(\mathcal{X}_d)$. By the Berkovich theorem, the suport of this colored polysimplicial complex equals $\Sigma(X_d)$.

Now, we will examine the strata to show that the corresponding cells can be seen in Γ_d. Consider a point in the special fiber of $\mathcal{X}_d$. It is given by a pair $(\mathcal{X}_0', D)$ satisfying the constraints from before. To this we may associate the pair $(G', \mathrm{mdeg} D)$, where G' is a subdivision of G and $(G', \mathrm{mdeg} D)$ if a stable pair over G.

The strata of $(\mathcal{X}_d)_0$ are exactly the loci where the dual pairs are constant. For example, the smooth locus of $(\mathcal{X}_d)_0$ is (X_0', D) where $X_0' \cong X_0$ and the strata are distinguished by mdegD. **We have an order preserving, 1-1 correspondence between the strata and the stable pairs**.

By examining the local equations, we can aslo show that the polysimplices get the correct colors, so we see that the two colored polysimplicial complexes are the same.