Computing Abstract Tropicalizations of Curves

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January 26, 2018

1 Introduction

Throughout the talk we let K be a field with valuation v, valuation ring R and characteristic 0, together with a uniformizer t.

Problem. Compute in an algorithmic way the *abstract tropicalization* of a smooth curve over K just from equations defining the curve. The main result that I will present in this talk is that we have found an algorithm for finding the abstract tropicalization of superelliptic curves.

First, I will explain the problem in more detail. Let C be a smooth curve over K. Given some equations defining the curve C, the coefficients of these equations will possibly involve the uniformizer t. Informally, we can think of the uniformizer t as "going to zero," and when $t = 0$, we will see some special and possibly singular behavior. Generically, the curve will be smooth, but it could limit to something singular. More formally, we need a *model* of the curve.

Definition 1. We say a *model* $\mathcal{C}$ for a curve C over K is a flat and finite type scheme over R whose generic fiber is isomorphic to C. We call this model *semistable* if the special fiber

$$\mathcal{C}_k = C \times_R k$$

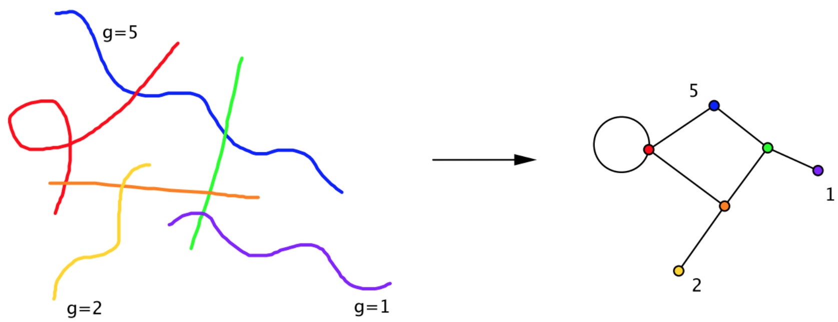
is a semistable curve over k, meaning:

1. It is reduced, connected, and only has nodal singularites;

2. every rational component has at least 2 singular points.

Definition 2. The *dual graph* of $\mathcal{C}_k$ has vertices corresponding to the irreducible components of $\mathcal{C}_k$, and edges corresponding to nodes.

Here is an example of a schematic of the special fiber of a curve C and on the right, its dual graph.



To define the *abstract tropicalization* of the curve C, we add a bit of extra data.

1. **Vertex Weights**: we add weights to the vertices by assigning to each vertex the *genus* of the corresponding component.

2. **Edge Lengths**: We add edge lengths in the following way. Given an edge corresponding to a node q between two components C_i and C_j, the completion of the local ring $\mathcal{O}_{\mathcal{C},q}$ is isomorphic to $R[[x, y]]/(xy - f)$ where $v(f) > 0$. Then, we define the length of the edge e_{ij} to be $v(f)$.

Theorem 1 ([Ber99], for the experts)**.** *The geometric realization of the dual graph is homeomorphic to the skeleton of the analytification. This is independent of any embedding of the curve C.*

So, why is this hard? Depending upon the equations that the curve arrives to you with, the singularities when $t = 0$ could be worse than nodes. By the *semistable reduction theorem*, we are always guaranteed in the abstract that we can replace a bad special fiber with a good one. However, this process is not algorithmic, and this proves to be the main difficulty in finding the abstract tropicalization of a curve in explicit examples.

The problem of computing the abstract tropicalization has been studied in several classes of curves.

1. In genus 1, the answer has been known for some time; one simply takes the valuation of the j-invariant of the curve and if it is negative, then the abstract tropicalization will be a cycle of length negative of this valuation.

2. The problem of computing the Berkovich skeleton for genus 2 curves was first studied in [**?**] in terms of semistable models. This was done systematically by studying the ramification data in [RSS14] and using Igusa invariants in [Hel16].

3. In the case of hyperelliptic curves, this problem was studied in [Cha13] and later solved in [BBCar] using ramification data and admissible covers. In [Hel17], Helminck presents criteria to reconstruct Berkovich skeleta using Laplacians on metric graphs. In this paper, we apply these techniques to the superelliptic case.

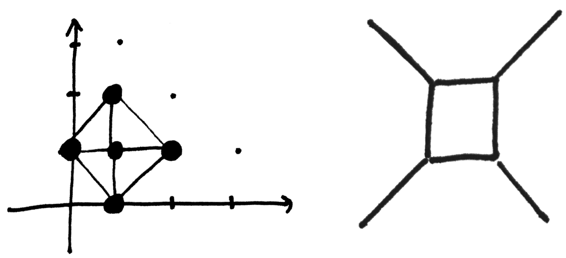
2 Embedded Tropicalization

A question one might ask is: how does this relate to the tropicalization of curves? Namely, if I have equations defining the curve, and I compute the embedded tropicalization of the curve, how does this relate to its abstract tropicalization? I will explain this and do an example to illustrate some of the issues for plane curves.

Example. Consider the plane curve over the Puiseux series in t defined by the equation

$$f(x, y, z) = txz^2 + tyz^2 + xyz + txy^2 + tx^2y.$$

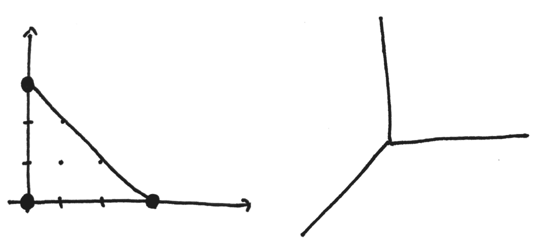
To find the embedded tropical hypersurface corresponding to this curve, we do the following. Make the Newton polygon, which is the convex hull of the exponent vectors. Then, find the regular subdivision induced from the weights given by the valuations of the coefficients. Then, the tropical curve will be dual to this (and rotated 180 degrees).



Now, we repeat this computation, but first we will do a change of coordinates. Consider

$$\begin{aligned} f(x - y - z, y - x - z, z + x + y) = &- x^3 - y^3 + z^3 \\ &+ \text{more middle terms} \\ &+ \text{terms higher order in } t. \end{aligned}$$

Then, we can compute the embedded tropicalization again.



We notice now that we have two very different answers, even though we were tropicalizing the same curve. However, some conclusions can be drawn:

Theorem 2 ([BPR16]). *Let C be a smooth curve of genus g in P^2_K. If the Newton polygon and subdivision corresponding to $Trop(C)$ form a unimodular triangulation, then the minimal skeleton of the tropicalization is isometric to the abstract tropicalization of C.*

So, we can conclude from the first tropicalization that the abstract tropicalization is a cycle of length equal to that of the length of the cycle in the embedded tropicalization.

So, we can use embedded tropicalization to find the abstract tropicalization of a curve– and this is called a *faithful* tropicalization. This can be thought of as a way of choosing a very nice embedding so that the embedded tropicalization reflects the geometry nicely.

There is a method for finding faithful tropicalizations called *tropical modifications*. This has been used to completely solve the problem in genus 1 and 2 (and possibly more...?)

We will take up a different approach in order to take advantage of information we know about the curve. This approach will allow us to compute abstract tropicalizations of superelliptic curves.

3 Superelliptic Curves

Definition 3. A *superelliptic curve* over K is a curve C which admits a Galois covering $\phi : C \to \mathbb{P}^1$ such that the Galois group is cyclic of order n. This is equivalent to the curve coming from an equation $y^n = f(x)$, where $f(x)$ can be assumed to be a polynomial. Then, the covering is given by $(x, y) \mapsto x$, and the ramification points are the roots of the polynomial f.

Our strategy for computing the abstract tropicalization of a superelliptic curve is as follows. The entire situation of the covering $C \to \mathbb{P}^1$ tropicalizes, meaning, that the abstract tropicalization of C will admit a map to the tree given by the tropicalization of $\mathbb{P}^1$ together with the marked ramification points. This map will be a nondegenerate harmonic morphism of graphs to a tree which is the quotient of a $\mathbb{Z}/n\mathbb{Z}$-action and satisfies the local Riemann-Hurwitz condition. From there, we can use some extra data about the divisor coming from the polynomial f to determine the missing details about the connectivity of the graph. Finally, the harmonicity of the cover and the local Riemann-Hurwitz conditions determine the edge lengths and vertex weights on the tropical curve.

I will now explain the algorithm by way of example.

Tropicalization Algorithm

Input: A curve C defined by the equation $y^n = f(x) = \prod_{i=1}^{r}(x - \alpha_i)$.
Output: The Abstract Tropicalization C_Σ of C.

Remark 1. For the input of this algorithm, we assume the function f has already been factored. Using the Newton-Puiseux Method [**?**], one can make a finite expansion for the roots. Since we are only interested in the valuations of the root differences, a finite expansion is sufficient. An explicit upper bound for the needed height of this expansion is given by $v(\Delta(f))$, where $\Delta(f)$ is the discriminant of f.

Example. We use as a running example finding the abstract tropicalization of the curve defined by the equation

$$y^3 = (x - 2)(x + t)(x - t)(x - 1 + t)(x - 1 - t + t^2)(x - 1 - t - t^2).$$

1. *Compute the tree T.* This is the abstract tropicalization of $\mathbb{P}^1$ together with the marked ramification points $Q_1, \ldots, Q_s$. This is done in the following way (See [MS15, Section 4.3]).

   Example. Let m_{ij} be the difference between the ith and the jth root:

   $$\begin{aligned} m = (&2+t, 2-t, 1+t, 1-t+t^2, 1-t-t^2, -2t, -1, -1-2t+t^2, \\ &-1-2t-t^2, -1+2t, -1+t^2, -1-t^2, -2t+t^2, -2t-t^2, -2t^2). \end{aligned}$$

   Let $d_{ij} = N - 2v(m_{ij})$, where v is the valuation on K and N is an integer such that $d_{ij} \geq 0$. Taking $N = 4$, we have $d = (4, 4, 4, 4, 4, 2, 4, 4, 4, 4, 4, 4, 2, 2, 0)$.

   The number d_{ij} is the distance between leaf i and leaf j in the tree T. These distances uniquely specify the tree T, and one can use the Neighbor Joining Algorithm [PS05, Algorithm 2.41] to reconstruct the tree T from these distances. Therefore, the tree is as displayed in Figure 1.

   Another way to think about this: blew up a tree of $\mathbb{P}^1$s to separate the roots in the special fiber. Dual to the tree. Each tree records the coefficient on t at that height of its t-adic expansion.

2. *Compute the slopes ψ_e along each edge of T.* There is a notion of divisors on metric graphs– finite formal linear combinations of points on the metric graph. There is also a notion of a principal divisor and equivalence of divisors which is analogous to those notions in algebraic geometry. The divisor $\text{div}(f)$ specializes to a principal divisor $\rho(\text{div}(f))$ is a principal divisor on T. Then, and so there exists a rational function $\psi : T \to \mathbb{R}$ on T so that the corresponding divisor is $\rho(\text{div}(f))$. One can compute $\rho(\text{div}(f))$ by observing where the zeros and poles of f specialize. Use this to compute the slopes ψ_e of ψ along edges e of T.

   Example. We have $\text{div}(f) = (2) + (-t) + (t) + (1-t) + (1+t-t^2) + (1+t+t^2) - 6\infty$ Then, $\rho(\text{div}(f)) = 2v_{56} + v_4 - 5v_1 + 2v_{23}$, and $\phi_{e_1} = 2, \phi_{e_2} = 3, \phi_{e_3} = 4$.

3. *Compute the intersection graph of $\mathcal{C}_s$.*

   (a) *Edges.* The number of preimages of each interior edge is $\gcd(\psi_e, n)$.

   (b) *Vertices.* The number of preimages of each vertex v is $\gcd(n, \psi_e | e \ni v)$.

   Example. Each of the edges e_1, e_3 has 1 preimage, all leaves have 1 preimage, and e_2 has 2 preimages. We can contract the leaves in the tropical curve, so we do not draw them in the graph, but we mention them here because they are necessary for bookkeeping the ramification in the formulas. Each of the vertices has one preimage.

4. *Determine the edge lengths and vertex weights to find C_Σ.*

Example. The lengths of all interior edges in the tree T were 1. In the graph, the edges will be scaled according to how many copies of them there are divided by n. So, th middle edge has length 1 and the outer two edges have length $1/3$. We can compute using the local RH formula that the outer two vertices have weight 1 and the middle vertices have weight 0.

So, the abstract tropicalization of our curve is the metric graph in Figure 2.

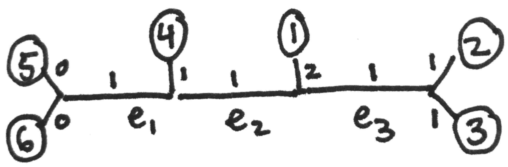


Figure 1: The tree T in Example 3.

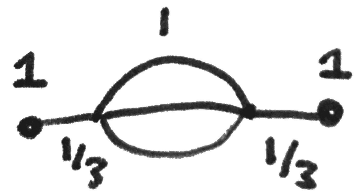


Figure 2: Tropicalization of the curve in Example 3.

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