Tropical Superelliptic Curves

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Abstract

Given a smooth curve defined over a valued field, it is a difficult problem to compute the Berkovich skeleton of the curve. In theory, one can find a semistable model for the curve and then find the dual graph of the special fiber, and this will give the skeleton. In practice, these procedures are not algorithmic and finding the model can become difficult. It is known how to find the Berkovich skeleton of genus one and genus two curves; more recently, the hyperelliptic case has also been solved. In this talk, we present the solution for superelliptic curves $y^n = f(x)$. This involves studying the covering from the curve to $\mathbb{P}^1$, and recovering data about the Berkovich skeleton from the tropicalization of $\mathbb{P}^1$ together with the marked ramification points. Throughout the talk we will study many examples in order to get a feel for the difficulties of this problem and how the procedure is carried out.

1 Introduction

Throughout the talk we let K be a field with valuation v and characteristic 0, together with a uniformizer π.

Problem. Compute in an algorithmic way the *abstract tropicalization* of a smooth curve over K just from equations defining the curve. The main result of the paper and this talk is that we have found an algorithm for finding the abstract tropicalization of superelliptic curves.

I will first give an informal description of the problem. We can think of the uniformizer π as "going to zero". Generically, the curve will be smooth, but it could limit to something singular. Then, we study the *dual graph* of the special fiber: to each irreducible component we associate a vertex, and to each node we associate an edge. We then weight each vertex by the *genus* of the corresponding component.

So, why is this hard? Depending upon the equations that the curve arrives to you with, the singularities when $\pi = 0$ could be worse than nodes. By the *semistable reduction theorem*, we are always guaranteed in the abstract that we can replace a bad special fiber with a good one. However, this process is not algorithmic, and this proves to be the main difficulty in finding the abstract tropicalization of a curve in explicit examples.

This problem has been studied in several classes of curves.

1. In genus 1, the answer has been known for some time; one simply takes the valuation of the j-invariant of the curve and if it is negative, then the abstract tropicalization will be a cycle of length negative of this valuation.
2. The problem of computing the Berkovich skeleton for genus 2 curves was first studied in [Liu93] in terms of semistable models. This was done systematically by studying the ramification data in [RSS14] and using Igusa invariants in [Hel16].
3. In the case of hyperelliptic curves, this problem was studied in [Cha13] and later solved in [BBCar] using ramification data and admissible covers. In [Hel17], Helminck presents criteria to reconstruct Berkovich skeleta using Laplacians on metric graphs. In this paper, we apply these techniques to the superelliptic case.

I will begin the talk with a more precise set up of the problem and some of the ideas we use. Then, I will present the algorithm found by my collaborator and I for computing the abstract tropicalization of a superelliptic curve. Lastly, I will provide some examples to illustrate the algorithm.

2 Setup of Problem

Let R be the valuation ring of K with maximal ideal m, let $k := R/m$ be the residue field, and let π be a uniformizer for K. Let C be a smooth curve over K.

Definition 1. A *superelliptic curve* over K is a curve C which admits a Galois covering $\phi : C \to \mathbb{P}^1$ such that the Galois group is cyclic of order n. Kummer theory [Neu99, Proposition 3.2] tells us the covering comes as $y^n = f(x)$, where $f(x)$ can be assumed to be a polynomial. Then, the covering is given by $(x, y) \mapsto x$, and the ramification points are the roots of the polynomial f.

Definition 2. We say a *model* $\mathcal{C}$ for a curve C over K is a flat and finite type scheme over R whose generic fiber is isomorphic to C. We call this model *semistable* if the special fiber

$$\mathcal{C}_k = C \times_R k$$

is a semistable curve over k, meaning that every smooth rational component meets the rest of the curve in at least two points, or every component has at least two points which are singular. By the semistable reduction theorem, our curve C always admits a semistable model.

The *dual graph* of $\mathcal{C}_k$ has vertices corresponding to the irreducible components of $\mathcal{C}_k$, and edges corresponding to nodes.

Theorem 1 ([Ber99], for the experts)**.** *The geometric realization of the dual graph is homeomorphic to the skeleton of the analytification. This is independent of any embedding of the curve C.*

Definition 3. To define the *abstract tropicalization* of the curve C, we add a bit of extra data. We add weights to the vertices by assigning to each vertex the *genus* of the corresponding component. We add edge lengths in the following way. Given an edge corresponding to a node q between two components C_i and C_j, the completion of the local ring $\mathcal{O}_{\mathcal{C},q}$ is isomorphic to $R[[x,y]]/(xy-f)$ where f is some element of the maximal ideal of R. Then, we define the length of the edge e_{ij} to be $v(f)$.

3 Galois Covers of Trees

Our strategy for computing the abstract tropicalization of a superelliptic curve is as follows. The entire situation of the covering $C \to \mathbb{P}^1$ tropicalizes, meaning, that the abstract tropicalization of C will admit a map to the tree given by the tropicalization of $\mathbb{P}^1$ together with the marked ramification points. This map will be a *superelliptic, nondegenerate harmonic morphism of graphs*. From there, we can use some extra data about the divisor coming from the polynomial f to determine the missing details about the connectivity of the graph. Finally, the harmonicity of the cover and the local Riemann-Hurwitz conditions determine the edge lengths and vertex weights on the tropical curve.

We first give the groundwork for discussing morphisms of metric graphs. In this setting, we assume some knowledge about metric graphs and models of metric graphs. We allow our graphs to have leaves of infinite length.

Definition 4. If (H,l) and (H',l') are loopless models for metric graphs Σ and Σ', then a *nondegenerate morphism of loopless models* $\theta : (H,l) \to (H',l')$ is a pair of maps $V(H) \to V(H')$ and $E(H) \to E(H')$ such that

1. If $e \in E(H)$ maps to $e' \in E(H')$, then the vertices of e must map to vertices of e'.
2. Infinite leaves in H map to infinite leaves in H'.
3. If $\theta(e) = e'$, then $l'(e')/l(e)$ is an integer. These must be specified if the edges have infinite length.

We say θ is *harmonic* if for every $v \in V(H)$, the *local degree*

$$d_v = \sum_{\substack{e \ni v,\\ \theta(e)=e'}} \frac{l'(e')}{l(e)}$$

is the same for all choices of $e' \in E(H')$.

As in [CMR16], we say that θ satisfies the *local Riemann-Hurwitz condition* at v if

$$2 - 2w(v) = d_v(2 - 2w'(\theta(v))) - \sum_{e \ni v} \left(\frac{l'(\theta(e))}{l(e)} - 1 \right).$$

Definition 5. An *automorphism* of Σ is a harmonic morphism $\theta : \Sigma \to \Sigma$ of degree 1. Given a subgroup G of $\mathrm{Aut}(\Sigma)$, the *quotient graph* Σ/G has a model H/G whose vertices are the G-orbits of $V(H)$ and whose edges are the G-orbits of edges defined by vertices lying in distinct G-orbits. If $\theta(e)$ is an edge in H/G, then $l(\theta(e)) = l(e) \cdot |\mathrm{Stab}(e)|$.

Definition 6. A nondegenerate harmonic morphism $\theta : \Sigma \to T$ is a *superelliptic* covering of metric graphs if θ is a quotient map with $G := \mathbb{Z}/n\mathbb{Z}$ and the target T is a tree, and θ satisfies the local Riemann-Hurwitz conditions at every point.

4 Statement of Algorithm and Example

Example. We use as a running example finding the abstract tropicalization of the curve defined by the equation

$$y^3 = x^2(x-\pi)(x-1)^2(x-1-\pi)(x-2)^2(x-2-\pi).$$

Tropicalization Algorithm

Input: A curve C defined by the equation $y^n = f(x) = \prod_{i=1}^{r}(x-\alpha_i)$.
Output: The Abstract Tropicalization C_Σ of C.

Remark 1. For the input of this algorithm, we assume the function f has already been factored. Using the Newton-Puiseux Method [Wal78], one can make a finite expansion for the roots. Since we are only interested in the valuations of the root differences, a finite expansion is sufficient. An explicit upper bound for the needed height of this expansion is given by $v(\Delta(f))$, where $\Delta(f)$ is the discriminant of f.

1. *Compute the tree* T. This is the abstract tropicalization of $\mathbb{P}^1$ together with the marked ramification points $Q_1, \dots, Q_s$. This is done in the following way (See [MS15, Section 4.3]).
   (a) Let M be the $2 \times s$ matrix whose columns are the branch points $Q_1, \dots, Q_s$. Let m_{ij} be the (i,j)-th minor of this matrix.
   (b) Let $d_{ij} = N - 2v(m_{ij})$, where v is the valuation on K and N is an integer such that $d_{ij} \geq 0$.
   (c) The number d_{ij} is the distance between leaf i and leaf j in the tree T. These distances uniquely specify the tree T, and one can use the Neighbor Joining Algorithm [PS05, Algorithm 2.41] to reconstruct the tree T from these distances.

Example. The matrix M is

$$M = \begin{bmatrix} 0 & \pi & 1 & 1+\pi & 2 & 2+\pi \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

and so the vector m (organized lexicographically) is

$$m = (-\pi,\ -1,\ -1-\pi,\ -2,\ -2-\pi,\ \pi-1,\ -1,\ \pi-2,\ -2,\ -\pi,\ -1,\ -1-\pi, -1+\pi,\ -1,\ -\pi).$$

Taking $N = 2$, we have $m = (0,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 0,\ 2,\ 2,\ 2,\ 2,\ 0)$. Therefore, the tree is as displayed in Figure 1.

2. *Compute the slopes ψ_e along each edge of T.* There is a notion of divisors on metric graphs– finite formal linear combinations of points on the metric graph. There is also a notion of a principal divisor and equivalence of divisors which is analogous to those notions in algebraic geometry. The divisor $\operatorname{div}(f)$ specializes to a principal divisor $\rho(\operatorname{div}(f))$ is a principal divisor on T. Then, and so there exists a rational function $\psi : T \to \mathbb{R}$ on T so that the corresponding divisor is $\rho(\operatorname{div}(f))$. One can compute $\rho(\operatorname{div}(f))$ by observing where the zeros and poles of f specialize. Use this to compute the slopes ψ_e of ψ along edges e of T.

   Example. We have $\operatorname{div}(f) = 2(0)+(\pi)+2(1)+(\pi+1)+2(2)+(2+\pi)-9(\infty)$. Then, $\rho(\operatorname{div}(f)) = 3v_{12}+3v_{34}+3v_{56}-9v$, and $\phi_{e_{12}} = \phi_{e_{34}} = \phi_{e_{56}} = 3$.

3. *Compute the intersection graph of $\mathcal{C}_s$.*

   (a) *Edges.* The number of preimages of each edge is $\gcd(\psi_e, n)$.

   (b) *Vertices.* The number of preimages of each vertex v is $\gcd(n, \psi_e | e \ni v)$.

   Example. Each of the edges e_{12}, e_{34}, e_{56} has 3 preimages, and all leaves have 1 preimage. We can contract these in the tropical curve, so we do not draw them in the graph, but we mention them here because they are necessary for bookkeeping the ramification in the formulas. The middle vertex v has 3 preimages, and the other vertices have 1 preimage. So, the graph is $K_{3,3}$.

4. *Determine the edge lengths and vertex weights to find C_Σ.*

   (a) *Edges.* If an edge e in T has length $l(e)$, then the length of each of its preimages in C_Σ is $\frac{l(e)\cdot\gcd(\psi_e,n)}{n}$. Remove any infinite leaf edges.

   (b) *Vertices.* The weight on each vertex v is determined by the local Riemann-Hurwitz formula. The degree d at v can be determined from definitions. The weight of v is determined by

$$2w(v) - 2 = -2\cdot d + \sum_{e\ni v}\left(\frac{n}{\gcd(n,\psi_e)} - 1\right).$$

   Example. The lengths of all interior edges in the tree T were 1. These lengths are preserved in $K_{3,3}$ because all edges were unramified. The weights on all vertices are 0. For example,

$$w(v_{12}) = -3 + 1 + (3(3/3-1) + 2(3/1) - 1)/2 = 0.$$

   So, the abstract tropicalization of our curve is the metric graph in Figure 2. Each vertex is labeled with its image in the tree T.

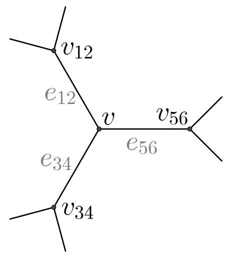

Figure 1: The tree T in Example 4.

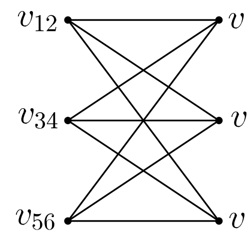

Figure 2: Tropicalization of the curve in Example 4.

5 Conclusion

In conclusion, in this paper we study the problem of computing the abstract tropicalization of a curve over a valued field. We provide an algorithm for computing this for superelliptic curves. Other main results include a theorem about realizability of superelliptic covers of metric graphs, and a computational study of the superelliptic locus inside the moduli space of tropical curves.

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