the Degree of SO(n)

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Introduction

This research was done with collaborators David Bruce, Taylor Brysiewicz, Robert Krone, and Elina Robeva.

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The group $\mathrm{SO}(n)$ is defined as

$$\mathrm{SO}(n) := \mathrm{SO}(n, \mathbb{C}) = \left\{ M \in \mathrm{Mat}_{n,n}(\mathbb{C}) \mid \det M = 1, \quad M^t M = \mathrm{Id} \right\}.$$

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Since these conditions are polynomials in the entries of the matrix M, we can view $\mathrm{SO}(n)$ as a complex variety.

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Why do we want to know this degree?

1. The degree is a fundamental piece of data about a variety that one would want to know about it, just like its dimension.
2. It provides the degree of low-rank semidefinite programming and is used to find the number of critical points for certain optimization problems.

Main Result

Theorem (B., Bruce, Brysiewicz, Krone, Robeva)

The degree of $\mathrm{SO}(n)$ *is given by:*

$$\deg \mathrm{SO}(n) = 2^{n-1} \det \left(\binom{2n-2i-2j}{n-2i}_{1 \leq i,j \leq \lfloor \frac{n}{2} \rfloor} \right).$$

Some Values

n	Symbolic	Numerical	Formula
2	2	2	2
3	8	8	8
4	40	40	40
5	384	384	384
6	-	4768	4768
7	-	111616	111616
8	-	-	3433600
9	-	-	196968448

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Gröbner bases methods were not as effective as numerical techniques using monodromy.

Kazarnovskij

Theorem (Kazarnovskij)

Let G be a connected reductive group of dimension m and rank r over an algebraically closed field. If $\rho : G \to \mathrm{GL}(V)$ is a representation with finite kernel then,

$$\deg \overline{\rho(G)} = \frac{m!}{|W(G)|(e_1!e_2!\cdots e_r!)^2|\ker(\rho)|}\int_{C_V}(\check{\alpha}_1\check{\alpha}_2\cdots\check{\alpha}_l)^2 dv.$$

where $W(G)$ is the Weyl group, e_i are Coxeter exponents, C_V is the convex hull of the weights, and $\check{\alpha}_i$ are the coroots.

$$\deg \mathrm{SO}(2r+1) = \frac{\binom{2r+1}{2}!}{r!2^r\prod_{k=1}^{r}(2k-1)!^2\cdot 1}\underbrace{\int_{C_V}\left(\prod_{1\le i<j\le r}(x_i^2-x_j^2)^2\prod_{i=1}^{r}(2x_i)^2\right)dv}_{I_{odd}}$$

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Proof of Main Theorem

We will only prove this result for $\mathrm{SO}(2r+1)$.

	$\mathrm{SO}(2r)$	$\mathrm{SO}(2r+1)$
Dimension	$\binom{2r}{2}$	$\binom{2r+1}{2}$
Rank	r	r
Positive Roots	$\{e_i \pm e_j\}_{i<j}$	$\{e_i \pm e_j\}_{i<j} \cup \{e_i\}$
Weights	$\{\pm e_i\}$	$\{\pm e_i\}$
$\lvert W(G) \rvert$	$r!2^{r-1}$	$r!2^r$
Coxeter Exponents	$1, 3, \ldots, 2r-3, r-1$	$1, 3, 5, \ldots, 2r-1$

Proof

$$I_{odd}(r) = \int_{C_v} \prod_{1 \le i < j \le r} (x_i^2 - x_j^2)^2 \prod_{i=1}^{r} (2x_i)^2 \, dv$$

Proof

$$I_{odd}(r) = \int_{C_v} \prod_{1 \le i < j \le r} (x_i^2 - x_j^2)^2 \prod_{i=1}^{r} (2x_i)^2 dv$$

$$= 2^r \int_{\Delta_r} \prod_{1 \le i < j \le r} (x_i^2 - x_j^2)^2 \prod_{i=1}^{r} (2x_i)^2 dv$$

Since the integrands are even in all variables, we can simplfy the region over which we integrate. Specifically, the integrals over C_V are 2^r times the same integrals over Δ_r, the standard r-simplex.

Proof

The next step is to turn the integrand into monomials. We use the well-known expression for the determinant of the Vandermonde matrix ($m_{i,j} = x_i^j$):

$$\prod_{1 \le i < j \le r} (y_j - y_i) = \sum_{\sigma \in S_r} \mathrm{sgn}(\sigma) \prod_{\mathrm{i}=1}^{\mathrm{r}} \mathrm{y}_{\mathrm{i}}^{\sigma(\mathrm{i})-1}.$$

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Substituting $y_i = x_i^2$ and squaring the entire expression yields

$$\prod_{1 \le i < j \le r} (x_i^2 - x_j^2)^2 = \sum_{\sigma,\tau \in S_r} \mathrm{sgn}(\sigma\tau) \prod_{i=1}^{r} x_i^{2\sigma(i)+2\tau(i)-4}.$$

Proof

$$I_{odd}(r) = \int_{C_v} \prod_{1 \le i < j \le r} (x_i^2 - x_j^2)^2 \prod_{i=1}^{r} (2x_i)^2 dv$$

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Proof

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\begin{aligned}
I_{odd}(r) &= \int_{C_v} \prod_{1\le i<j\le r} (x_i^2 - x_j^2)^2 \prod_{i=1}^{r} (2x_i)^2 dv \\
&= 2^r \int_{\Delta_r} \prod_{1\le i<j\le r} (x_i^2 - x_j^2)^2 \prod_{i=1}^{r} (2x_i)^2 dv \\
&= 2^r \int_{\Delta_r} \left(\sum_{\sigma,\tau\in S_r} \mathrm{sgn}(\sigma\tau) \prod_{\mathrm{i}=1}^{\mathrm{r}} \mathrm{x}_{\mathrm{i}}^{2\sigma(\mathrm{i})+2\tau(\mathrm{i})-4} \right) \prod_{i=1}^{r} (2x_i)^2 dv
\end{aligned}
$$

Expanding the binomials in the way described on the previous slide.

Proof

$$
\begin{aligned}
I_{odd}(r) &= \int_{C_v} \prod_{1\leq i<j\leq r} (x_i^2 - x_j^2)^2 \prod_{i=1}^{r} (2x_i)^2 dv \\
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&= 2^{3r} \sum_{\sigma,\tau\in S_r} \mathrm{sgn}(\sigma\tau) \int_{\Delta_r} \left(\prod_{i=1}^{r} x_i^{2\sigma(i)+2\tau(i)-2} \right) dv.
\end{aligned}
$$

Proof

Now, we apply the following proposition for integrating monomials over a simplex.

Proposition (Found in Algebraic Number Theory)

Let $\Delta_r \subset \mathbb{R}^r$ be the standard r-simplex. If $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r_{>0}$ then

$$\int_{\Delta_r} x^{\mathbf{a}} d\mathbf{x} = \frac{1}{(\sum a_i + r)!} \prod_{i=1}^{r} a_i!.$$

Proof

$$I_{odd}(r) = 2^{3r} \sum_{\sigma,\tau \in S_r} \mathrm{sgn}(\sigma\tau) \int_{{}'_{\mathrm{r}}} \left(\prod_{\mathrm{i}=1}^{\mathrm{r}} \mathrm{x}_{\mathrm{i}}^{2\sigma(\mathrm{i})+2\tau(\mathrm{i})-2} \right) \mathrm{dv}$$

Proof

$$I_{odd}(r) = 2^{3r} \sum_{\sigma,\tau \in S_r} \operatorname{sgn}(\sigma\tau) \int_{\prime_r} \left(\prod_{i=1}^{r} x_i^{2\sigma(i)+2\tau(i)-2} \right) dv$$

$$= \frac{2^{3r}}{\left(4\binom{r}{2} + 3r\right)!} \sum_{\sigma,\tau \in S_r} \operatorname{sgn}(\sigma\tau) \prod_{i=1}^{r} (2\sigma(i) + 2\tau(i) - 2)!$$

Using the previous slide and the fact that the integrand is homogeneous of degree $4\binom{r}{2} + 2r$.

Proof

$$
\begin{aligned}
I_{odd}(r) &= 2^{3r} \sum_{\sigma,\tau \in S_r} \operatorname{sgn}(\sigma\tau) \int_{\acute{}\,\mathrm{r}} \left(\prod_{\mathrm{i}=1}^{\mathrm{r}} \mathrm{x}_{\mathrm{i}}^{2\sigma(\mathrm{i})+2\tau(\mathrm{i})-2} \right) \mathrm{dv} \\
&= \frac{2^{3r}}{\left(4\binom{r}{2} + 3r\right)!} \sum_{\sigma,\tau \in S_r} \operatorname{sgn}(\sigma\tau) \prod_{\mathrm{i}=1}^{\mathrm{r}} (2\sigma(\mathrm{i}) + 2\tau(\mathrm{i}) - 2)! \\
&= \frac{2^{3r}}{\left(4\binom{r}{2} + 3r\right)!} \sum_{\sigma,\tau \in S_r} \operatorname{sgn}(\sigma\tau) \prod_{\mathrm{i}=1}^{\mathrm{r}} (2\mathrm{i} + 2\tau\sigma^{-1}(\mathrm{i}) - 2)!
\end{aligned}
$$

Replacing i with $\sigma^{-1}(i)$.

Proof

$$I_{odd}(r) = \frac{2^{3r}}{\left(4\binom{r}{2} + 3r\right)!} \sum_{\sigma,\tau \in S_r} \mathrm{sgn}(\sigma\tau) \prod_{i=1}^{r} (2i + 2\tau\sigma^{-1}(i) - 2)!$$

Proof

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$$= \frac{r!\,2^{3r}}{\left(4\binom{r}{2} + 3r\right)!} \sum_{\rho \in S_n} \operatorname{sgn}(\rho) \prod_{i=1}^{r} (2i + 2\rho(i) - 2)!$$

Let $\rho = \tau\sigma^{-1}$. Over all pairs $\sigma, \tau \in S_r$, permutation ρ takes the value of each permutation in S_r exactly $r!$ times, and $\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\rho)$.

Proof

$$I_{odd}(r) = \frac{2^{3r}}{\left(4\binom{r}{2} + 3r\right)!} \sum_{\sigma,\tau \in S_r} \operatorname{sgn}(\sigma\tau) \prod_{i=1}^{r} (2i + 2\tau\sigma^{-1}(i) - 2)!$$

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$$= \frac{r!2^{3r}}{\left(4\binom{r}{2} + 3r\right)!} \det\left((2i + 2j - 2)!\right)_{1 \le i,j \le n}.$$

We recognize this as a determinant.

Proof

$$
\begin{aligned}
I_{odd}(r) &= \frac{2^{3r}}{\left(4\binom{r}{2}+3r\right)!}\sum_{\sigma,\tau\in S_r}\mathrm{sgn}(\sigma\tau)\prod_{\mathrm{i}=1}^{\mathrm{r}}(2\mathrm{i}+2\tau\sigma^{-1}(\mathrm{i})-2)! \\
&= \frac{r!2^{3r}}{\left(4\binom{r}{2}+3r\right)!}\sum_{\rho\in S_n}\mathrm{sgn}(\rho)\prod_{\mathrm{i}=1}^{\mathrm{r}}(2\mathrm{i}+2\rho(\mathrm{i})-2)! \\
&= \frac{r!2^{3r}}{\left(4\binom{r}{2}+3r\right)!}\det\left((2i+2j-2)!\right)_{1\le i,j\le n}.
\end{aligned}
$$

The main theorem now follows directly from some simplification, by pushing factorial factors from the denominator in to the rows and columns of the matrix to make binomial coefficients.

Non-Intersecting Lattice Paths

The formula given in the main theorem can be interpreted as a count of non-intersecting lattice paths via the Gessel-Viennot Lemma [GV85].

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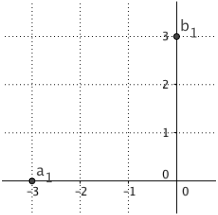
Lemma (Gessel-Viennot (Weak Version))

Let $A = \{a_1, \ldots, a_r\}, B = \{b_1, \ldots, b_r\} \subseteq \mathbb{Z}^2$. *Let* $M_{i,j}$ *be the number of North-East lattice paths from* a_i *to* b_j. *If the only way that a system of North-East lattice paths from* $A \to B$ *do not cross each other is by sending* $a_i \mapsto b_i$ *then the determinant of* M *is given by the number of such non-intersecting lattice paths.*

Example: SO(5)

We will count all north-east lattice paths $a_1 \to b_1$.



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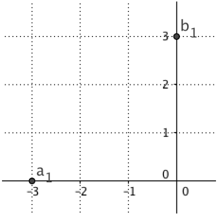

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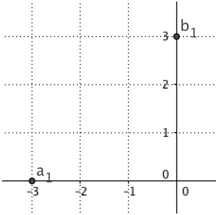


Since the total number of steps taken is 6 and 3 of these must be eastward, we have that the total number of paths is $\binom{6}{3}$.
If we add two more points at $(-1, 0)$ and $(0, 1)$, the matrix M in this case is:

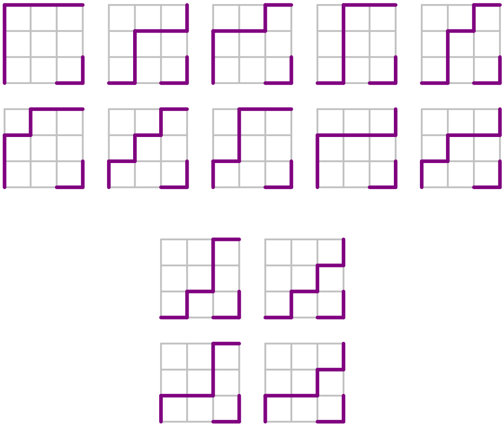
$$\begin{bmatrix} \binom{6}{3} & \binom{4}{1} \\ \binom{4}{3} & \binom{2}{1} \end{bmatrix}.$$

Example: SO(5)

Its determinant is 24, which is $\deg(SO(5))/2^4$.

Example: SO(5)

Its determinant is 24, which is $\deg(SO(5))/2^4$. Here are 14 of the 24 paths. The missing 10 are obtained by taking the first 10 in the picture, and "flipping" the lower right path.



$\deg(\mathrm{SO}(n))$ in Terms of Lattice Paths

The determinant appearing in the degree of $\mathrm{SO}(n)$ has a natural interpretation via Gessel-Viennot because binomial coefficients count lattice paths.

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Theorem (B., Bruce, Brysiewicz, Krone, Robeva)

$\deg \mathrm{SO}(n) = 2^{n-1}(\#\{\textit{Non-Intersecting Lattice Paths from } A \textit{ to } B\})$,

where the positions of A and B are given by
$a_i = (2i-n, 0)$, $b_j = (0, n-2j)$ *where* $1 \leq i, j \leq \lfloor \frac{n}{2} \rfloor$.

$$\deg \mathrm{SO}(n) = 2^{n-1} \det \left(\binom{2n-2i-2j}{n-2i}_{1 \leq i,j \leq \lfloor \frac{n}{2} \rfloor} \right).$$

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