The slack realization space of a matroid

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Goal

Goal: Make spaces whose points correspond to realizations of a matroid, and study that space to learn about the matroid.

Why: We can perform computations on the space, and these computations can quickly answer questions about the matroid:

- Realizability
- Projective Uniqueness

Matroids

Matroids are well studied objects which provide a combinatorial abstraction of linear independence in vector spaces.

Definition

A **rank $d+1$ Matroid on n elements** is a subset $\mathcal{B}$ of $\binom{\{1,\dots,n\}}{d+1}$ called the **bases** of the matroid, satisfying:

- $\mathcal{B}$ is nonempty,
- If $A, B \in \mathcal{B}$ and $a \in A \backslash B$ then there exists $b \in B \backslash A$ such that $A \backslash \{a\} \cup \{b\} \in \mathcal{B}$.

Realizable Matroids

Given a vector space V over a field k and vectors $v_1, \dots, v_n \in V$ spanning V, the collection of subsets of $\{1, \dots, n\}$ indexing bases of V gives a matroid which we denote $M[V]$.

Such a matroid is called **realizable over k**, and $v_1, \dots, v_n$ are called a **realization**.

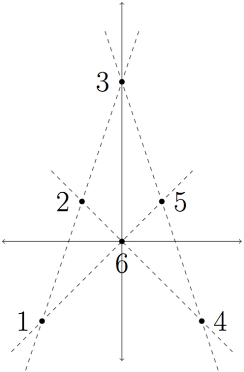
There are examples of matroids which are not realizable. This depends very much on the field.

Example

Consider the rank 3 matroid $M[V]$ for V whose vectors are

$$v_1 = (-2, -2, 1), \quad v_2 = (-1, 1, 1),$$
$$v_3 = (0, 4, 1), \quad v_4 = (2, -2, 1),$$
$$v_5 = (1, 1, 1), \quad v_6 = (0, 0, 1).$$

Projecting onto the plane $z = 1$, this can be visualized as the points of intersection of four lines in the plane.



Slack matrix

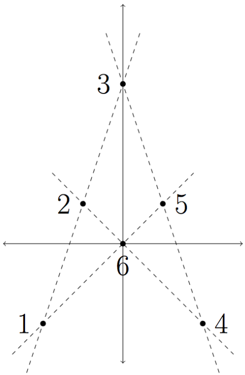
Let $M = M[V]$ be a realizable matroid with realization V. The **hyperplanes** of the matroid are collections of the $v_1, \ldots, v_n$ which are contained in a subspace of dimension d.

Definition

The **slack matrix** of the matroid $M = M[V]$ over k is the $n \times h$ matrix $S_M = V^\top W$, where

- W is the matrix whose columns are the hyperplane defining normals,
- V is the matrix with columns $v_1, \ldots, v_n$.

Slack matrix: Example



$$\begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array}\begin{bmatrix} -2 & -2 & 1\\ -1 & 1 & 1\\ 0 & 4 & 1\\ 2 & -2 & 1\\ 1 & 1 & 1\\ 0 & 0 & 1 \end{bmatrix} \begin{array}{ccccccc} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7\\ 123 & 246 & 345 & 156 & 25 & 14 & 36 \end{array} \begin{bmatrix} -3 & 3 & 6 & -3 & 0 & 0 & 4\\ 1 & 3 & 2 & 3 & 2 & 4 & 0\\ -4 & 0 & -8 & 0 & -2 & 8 & 0 \end{bmatrix} =$$

$$\begin{array}{c} \\ \\ 1\\2\\3\\4\\5\\6 \end{array} \begin{array}{ccccccc} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7\\ 123 & 246 & 345 & 156 & 25 & 14 & 36 \end{array}$$

$$\begin{bmatrix} 0 & -12 & -24 & 0 & -6 & 0 & -8\\ 0 & 0 & -12 & 6 & 0 & 12 & -4\\ 0 & 12 & 0 & 12 & 6 & 24 & 0\\ -12 & 0 & 0 & -12 & -6 & 0 & 8\\ -6 & 6 & 0 & 0 & 0 & 12 & 4\\ -4 & 0 & -8 & 0 & -2 & 8 & 0 \end{bmatrix}$$

Properties of Slack Matrices

Here are some of the important properties of slack matrices.

Lemma

The rows of a slack matrix S_M form a realization of the matroid M.

Theorem (B-Wiebe)

A matrix $S \in k^{n \times h}$ is the slack matrix of some realization of M if and only if both of the following hold:

1. $supp(S) = supp(S_{M[V]})$
2. $rank(S) = d + 1.$

These are algebraic conditions on the entries of the matrix.

The Slack Ideal

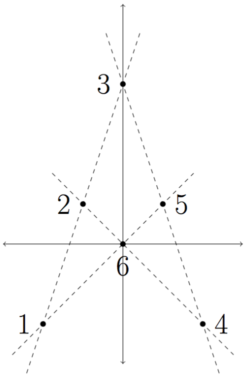
The **symbolic slack matrix** of matroid M is the matrix $S_M(\mathbf{x})$ with rows indexed by elements $i \in E$, columns indexed by hyperplanes $H_j \in \mathcal{H}(M)$ and (i, j)-entry

$$\begin{cases} x_{ij} & \text{if } i \notin H_j \\ 0 & \text{if } i \in H_j. \end{cases}$$

The **slack ideal** of M is the saturation of the ideal generated by the $(d+2)$-minors of $S_M(\mathbf{x})$, namely

$$I_M := \Big\langle (d+2) - \text{minors of } S_M(\mathbf{x}) \Big\rangle : \left(\prod_{i=1}^{n} \prod_{j : i \notin H_j} x_{ij} \right)^{\infty} \quad \subset k[\mathbf{x}].$$

Example



	H_1	H_2	H_3	H_4	H_5	H_6	H_7
	123	246	345	156	25	14	36
1	0	x_{12}	x_{13}	0	x_{15}	0	x_{17}
2	0	0	x_{23}	x_{24}	0	x_{26}	x_{27}
3	0	x_{32}	0	x_{34}	x_{35}	x_{36}	0
4	x_{41}	0	0	x_{44}	x_{45}	0	x_{47}
5	x_{51}	x_{52}	0	0	0	x_{56}	x_{57}
6	x_{61}	0	x_{63}	0	x_{65}	x_{66}	0

Now, we take the 4×4 minors and saturate...

Example

There are 72 binomial generators of its slack ideal:

deg 2	$x_{36}x_{65}+x_{35}x_{66},\ x_{26}x_{63}-x_{23}x_{66},\ x_{15}x_{63}-x_{13}x_{65},\ x_{56}x_{61}-x_{51}x_{66},\ x_{45}x_{61}-x_{41}x_{65},$ $x_{27}x_{56}+x_{26}x_{57},\ x_{36}x_{52}-x_{32}x_{56},\ x_{17}x_{52}-x_{12}x_{57},\ x_{47}x_{51}-x_{41}x_{57},\ x_{17}x_{45}+x_{15}x_{47},$ $x_{35}x_{44}-x_{34}x_{45},\ x_{27}x_{44}-x_{24}x_{47},\ x_{26}x_{34}-x_{24}x_{36},\ x_{15}x_{32}-x_{12}x_{35},\ x_{17}x_{23}-x_{13}x_{27}$
deg 3	$x_{47}x_{56}x_{65}-x_{45}x_{57}x_{66},\ x_{17}x_{56}x_{65}+x_{15}x_{57}x_{66},\ x_{12}x_{56}x_{65}+x_{15}x_{52}x_{66},\ x_{26}x_{47}x_{65}+x_{27}x_{45}x_{66},$ $x_{26}x_{44}x_{65}+x_{24}x_{45}x_{66},\ x_{17}x_{26}x_{65}-x_{15}x_{27}x_{66},\ x_{17}x_{56}x_{63}+x_{13}x_{57}x_{66},\ x_{12}x_{56}x_{63}+x_{13}x_{52}x_{66},$ $x_{27}x_{45}x_{63}+x_{23}x_{47}x_{65},\ x_{24}x_{45}x_{63}+x_{23}x_{44}x_{65},\ x_{12}x_{36}x_{63}+x_{13}x_{32}x_{66},\ x_{24}x_{35}x_{63}+x_{23}x_{34}x_{65},$ $x_{23}x_{57}x_{61}+x_{27}x_{51}x_{63},\ x_{15}x_{57}x_{61}+x_{17}x_{51}x_{65},\ x_{13}x_{57}x_{61}+x_{17}x_{51}x_{63},\ x_{35}x_{52}x_{61}+x_{32}x_{51}x_{65},$ $x_{15}x_{52}x_{61}+x_{12}x_{51}x_{65},\ x_{13}x_{52}x_{61}+x_{12}x_{51}x_{63},\ x_{26}x_{47}x_{61}+x_{27}x_{41}x_{66},\ x_{23}x_{47}x_{61}+x_{27}x_{41}x_{63},$ $x_{13}x_{47}x_{61}+x_{17}x_{41}x_{63},\ x_{36}x_{44}x_{61}+x_{34}x_{41}x_{66},\ x_{26}x_{44}x_{61}+x_{24}x_{41}x_{66},\ x_{23}x_{44}x_{61}+x_{24}x_{41}x_{63},$ $x_{35}x_{47}x_{56}+x_{36}x_{45}x_{57},\ x_{34}x_{47}x_{56}+x_{36}x_{44}x_{57},\ x_{17}x_{35}x_{56}-x_{15}x_{36}x_{57},\ x_{35}x_{47}x_{52}+x_{32}x_{45}x_{57},$ $x_{34}x_{47}x_{52}+x_{32}x_{44}x_{57},\ x_{27}x_{34}x_{52}+x_{24}x_{32}x_{57},\ x_{13}x_{26}x_{52}+x_{12}x_{23}x_{56},\ x_{36}x_{45}x_{51}+x_{35}x_{41}x_{56},$ $x_{32}x_{45}x_{51}+x_{35}x_{41}x_{52},\ x_{12}x_{45}x_{51}+x_{15}x_{41}x_{52},\ x_{36}x_{44}x_{51}+x_{34}x_{41}x_{56},\ x_{32}x_{44}x_{51}+x_{34}x_{41}x_{52},$ $x_{26}x_{44}x_{51}+x_{24}x_{41}x_{56},\ x_{27}x_{36}x_{45}-x_{26}x_{35}x_{47},\ x_{17}x_{32}x_{44}+x_{12}x_{34}x_{47},\ x_{15}x_{23}x_{44}+x_{13}x_{24}x_{45},$ $x_{17}x_{26}x_{35}+x_{15}x_{27}x_{36},\ x_{13}x_{26}x_{35}+x_{15}x_{23}x_{36},\ x_{15}x_{27}x_{34}+x_{17}x_{24}x_{35},\ x_{15}x_{23}x_{34}+x_{13}x_{24}x_{35},$ $x_{17}x_{26}x_{32}+x_{12}x_{27}x_{36},\ x_{13}x_{26}x_{32}+x_{12}x_{23}x_{36},\ x_{17}x_{24}x_{32}+x_{12}x_{27}x_{34},\ x_{13}x_{24}x_{32}+x_{12}x_{23}x_{34}$
deg 4	$x_{27}x_{35}x_{52}x_{63}-x_{23}x_{32}x_{57}x_{65},\ x_{17}x_{36}x_{44}x_{63}-x_{13}x_{34}x_{47}x_{66},\ x_{24}x_{35}x_{57}x_{61}-x_{27}x_{34}x_{51}x_{65},$ $x_{23}x_{34}x_{52}x_{61}-x_{24}x_{32}x_{51}x_{63},\ x_{12}x_{36}x_{47}x_{61}-x_{17}x_{32}x_{41}x_{66},\ x_{13}x_{32}x_{44}x_{61}-x_{12}x_{34}x_{41}x_{63},$ $x_{15}x_{26}x_{44}x_{52}-x_{12}x_{24}x_{45}x_{56},\ x_{13}x_{26}x_{45}x_{51}-x_{15}x_{23}x_{41}x_{56},\ x_{12}x_{23}x_{44}x_{51}-x_{13}x_{24}x_{41}x_{52}$

Slack Realization Space

Suppose there are t variables in $S_M(\mathbf{x})$. The **slack variety** is the variety $\mathcal{V}(I_M) \subset k^t$.

Theorem (B-Wiebe)

Let M be a rank $d+1$ matroid. Then V is a realization of M if and only if $S_{M[V]} \in \mathcal{V}(I_M) \cap (k^*)^t$.

Non-Realizability

We now discuss how the slack variety can be used to study realizability of the matroid.

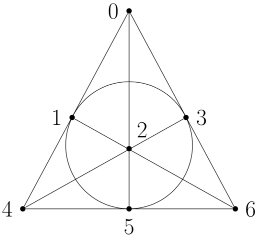
Theorem (B-Wiebe)

A matroid M has a realization over k if and only if $\mathcal{V}(I_M) \cap (K^*)^t$ is nonempty.

In other words, if the slack ideal $I_M = \langle 1 \rangle$ over k, then M is not realizable over k. If k is algebraically closed and M is not realizable over k, then $I_M = \langle 1 \rangle$.

Non-Realizability: Example

We now study the Fano matroid, whose nonbases are lines and circle below. It has 7 hyperplanes given by the collinear triples and the circle. It is known to only be realizable over characteristic 2.



	H_1	H_2	H_3	H_4	H_5	H_6	H_7
	126	014	456	025	036	234	135
0	x_{01}	0	x_{03}	0	0	x_{06}	x_{07}
1	0	0	x_{13}	x_{14}	x_{15}	x_{16}	0
2	0	x_{22}	x_{23}	0	x_{25}	0	x_{27}
3	x_{31}	x_{32}	x_{33}	x_{34}	0	0	0
4	x_{41}	0	0	x_{44}	x_{45}	0	x_{47}
5	x_{51}	x_{52}	0	0	x_{55}	x_{56}	0
6	0	x_{62}	0	x_{64}	0	x_{66}	x_{67}

Over $\mathbb{Q}$, one may verify in *Macaulay2* that the slack ideal $I_M = \langle 1 \rangle$. Over $\mathbb{F}_2$, we find that the slack ideal is generated by 126 binomials, and that the all ones slack matrix is the only point on this variety.

Projective Uniqueness

We say two realizations V and V' of a matroid M are projectively equivalent if $V' = AVB$ for some $A \in GL(k^{d+1})$ and B is a k^*-multiple of a permutation matrix.

Lemma

Two realizations of a matroid M are projectively equivalent if and only if their slack matrices are the same up to row and column scaling.

So, the slack variety is closed under the action of the torus $(k^*)^n \times (k^*)^h$, which acts by row and column scaling.

Projective Uniqueness

We can select an element of a projective equivalence class in the following way.

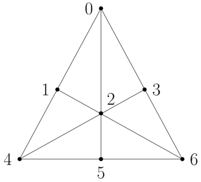
Lemma

Let T be a maximal tree in the bipartite non-incidence graph of the matroid. Given a realization of M and its slack matrix S_M, we can always row and column scale S_M to have ones in the entries corresponding to edges of T.

Then, we obtain the **scaled slack ideal**.

Projective Uniqueness: Example

We now consider the non-Fano matroid.



	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9
	126	014	456	025	036	234	35	13	15
0	x_{01}	0	x_{03}	0	0	1	x_{07}	1	x_{09}
1	0	0	1	x_{14}	x_{15}	1	0	0	0
2	0	1	x_{23}	0	x_{25}	0	x_{27}	x_{28}	x_{29}
3	x_{31}	1	x_{33}	x_{34}	0	0	0	0	x_{39}
4	1	0	0	x_{44}	x_{45}	0	x_{47}	x_{48}	x_{49}
5	1	1	0	0	1	1	0	x_{58}	0
6	0	x_{62}	0	1	0	1	1	x_{68}	1

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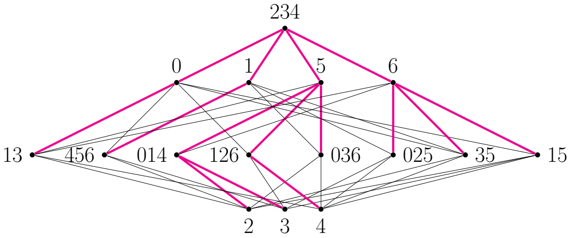

Projective Uniqueness: Example

We compute over $\mathbb{Q}$ in *Macaulay2* that the scaled slack ideal consists entirely of linear equations, and the scaled slack variety contains a single point:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & -1 & 1 \end{pmatrix}.$$

So, the nonfano matroid is projectively unique over $\mathbb{Q}$.

Conclusion

In this talk, we:

1. Made a new realization space for matroids
2. Demonstrate how to use this space to test for realizability
3. Demonstrate how to use this to test for projective uniqueness

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Thank You