Simplicial Complexes

Madeline Brandt

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Definition 1. We say points $v_0, \ldots, v_n \in \mathbb{R}^k$ are **affinely independent** if they span an affine n-plane. Another way to view this is that no point is contained in the convex hull of the other points.

Definition 2. If $v_0, \ldots, v_n$ are affinely independent then the **affine simplex** σ spanned by the v_i is

$$\sigma = (v_0, \ldots, v_n) = \left\{ \sum_{i=0}^{n} \lambda_i v_i \ \middle| \ \sum_{i=0}^{n} \lambda_i = 1, \lambda_i \geq 0 \right\}.$$

This is just the convex hull of the points v_i. A k-**face** of σ is any affine simplex spanned by a subset of size $k+1$ of the v_i.

Definition 3. A (geometric) **simplicial complex** K is a collection of affine simplices such that

1. If $\sigma \in K$ then any face of σ is in K.

2. If σ and τ are in K, then $\sigma \cap \tau$ is a face of both σ and τ, or is empty.

Example. The following give an example and a nonexample of a simplicial complex.

If K is a simplicial complex then we call $|K| = \cup \{\sigma \mid \sigma \in K\}$ the **support** of K.

Definition 4. A space X is a **triangulatable** if there exists a homeomorphism from $|K|$ to X for some simplicial complex K. The homeomorphism together with K is called a **triangulation** of X.

Example. The following gives a triangulation of the real projective plane. This agrees with the triangulation given by `Polymake`.
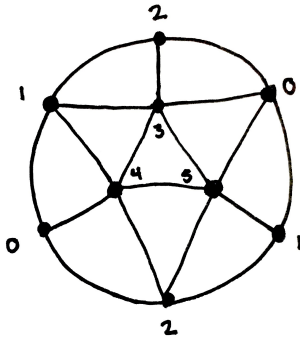

```
polytope > application "topaz";
topaz > $r = real_projective_plane();
topaz > print $r->FACETS;
0 1 4
0 1 5
0 2 3
0 2 4
0 3 5
1 2 3
1 2 5
1 3 4
2 4 5
3 4 5
```

The simplicial complex structure on K gives a cell complex structure on $|K|$: the cells are the simplices, with characteristic maps from the simplex into $|K|$. In this setting, we get a chain complex whose nth group is free abelian on the n-simplices of K.

The **boundary formula** of cellular homology reduces to the following:

$$\partial < v_{\sigma_0}, \ldots, v_{\sigma_n} > = \sum_{i=0}^{n} (-1)^i < v_{\sigma_0}, \ldots \hat{v}_{\sigma_i}, \ldots, v_{\sigma_n} > .$$

The reason is because the degree $[\tau : \sigma]$ will be 0 unless τ is a face of σ, and otherwise is is degree ± 1. Since we require $\partial \partial = 0$, we must have that the signs alternate.

By considering this as a CW-complex, we have shown that the homology of this chain complex is equal to cellular homology.

Simplicial homology can be cumbersome in the sense that there are usually many more simplices required than you would need cells to do cellular homology. The real projective plane requires one cell of each dimension $0, 1$, and 2 for cellular homology, but for simplicial homology one needs 6 vetices, 15 edges and 10 faces. The advantage is that the boundary maps are very mechanical to compute.

Example. Homology of circle. This comes to us as a triangle. The chain complex is given by

$$\cdots 0 \to \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z}^3 \xrightarrow{\partial_0} 0$$

Let the basis for Δ_1 be $\{e_1, e_2, e_1 - e_2 + e_3\}$, and the basis for Δ_0 be $\{v_3 - v_2, v_3 - v_1, v_3\}$
We now compute the maps ∂_0, ∂_1:

$$\partial_0 = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \qquad \partial_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then $H_0(S^1) = \mathbb{Z}$, and $H_1(S^1)$ is $\mathbb{Z}^3/\mathbb{Z}^2 = \mathbb{Z}$. All other homologies will be 0.

Example. Computation of simplicial homology for $\mathbb{RP}^2$. Given the simplicial complex showed above, we have the following chain complex:

$$\cdots 0 \to \mathbb{Z}^{10} \xrightarrow{\partial_2} \mathbb{Z}^{15} \xrightarrow{\partial_1} \mathbb{Z}^6 \xrightarrow{\partial_0} 0$$

We can compute the boundary maps by hand, or we can use `Macaulay2` to compute the boundary maps. `Macaulay2` has implemented Gröbner Basis methods, along with a correspondence between simplicial complexes and squarefree monomial ideals, to compute chain complexes for simplicial complexes and their homologies. To do this, we make a variable for each vertex $0 \mapsto$ `a`, ..., $5 \mapsto$ `f`, and add one monomial for each facet.

```
i1 :  loadPackage "SimplicialComplexes"
i2 :  R = ZZ[a..f]
i3 :  D = simplicialComplex monomialIdeal(a*b*e, a*b*f, a*c*d, a*c*e,
a*d*f, b*c*d, b*c*f, b*d*e, c*e*f, d*e*f)
```

Of course, the map ∂_0 is the 0 map. `Macaulay2` tells us that the map ∂_1 is

$$\partial_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}$$

The Smith Normal Form of this matrix has 5 ones on the diagonal, which implies that $H_0(\mathbb{RP}^2) = \mathbb{Z}$. This was done using the code

```
i4 :  c = chainComplex D
i5 :  c.dd_1
i6 :  M1=smithNormalForm c.dd_1
i7 :  M1_0
```

`Macaulay2` tells us that the map ∂_2 is

$$\partial_2 = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

The Smith Normal Form of this matrix has nine 1's on the diagonal and one 2. This means that $H_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$. This also tells us that the map is injective, and so $H_1(\mathbb{RP}^2) = 0$.

We can verify this computation in `Polymake` (which does reduced homology):

```
topaz > print $r->HOMOLOGY;
( 0)
((2 1) 0)
( 0)
```

The output is interpreted in the following way: The first line give the 0th homology group to be 0. The second line says that the first homology has one 2-torsion component, and no $\mathbb{Z}$ component. The third line says that there is no second homology.

How to use `Polymake`:

We explain the code used in the example for $\mathbb{RP}^2$. For more information about `Polymake` and `Topaz`, go to

https://polymake.org/doku.php/release_docs/latest/topaz.html

Polymake code:	Explanation:
`application "topaz";`	Polymake has different "applications," `Topaz` does topology things. Always need a ; before entering
`$s = real_projective_plane();`	variables always start with $, `real_projective_plane()` summons a simplicial complex for $\mathbb{RP}^2$
`print $s->HOMOLOGY;`	"properties" of an object are always in all caps,
`print $s-> FACETS;`	to see them you type `print $s->PROPERTY_NAME`.