POLYTOPES

Madeline Brandt

April 8, 2016

This lecture will be primarily based on chapter 5 of the celebrated book by Günter Ziegler: *Lectures on Polytopes* [Zie95]. We motivate this choice with the following quotation by Chvátal [Chv83]:

> Here, however, a word of warning may be in order: do *not* try to visualize n-dimensional objects for $n \geq 4$. Such an effort is not only doomed to failure– it may be dangerous to your mental health. (If you do succeed, then you are in trouble.)

Today we will try to visualize $n = 4$ dimensions, despite the above recommendation. The goal for this talk is to understand what a polytope and some related constructions are. We also wish to understand some tools for examining polytopes; in particular, we will give detailed examples of Schlegel diagrams of four dimensional polytopes. First, we give some necessary definitions.

Definitions

We begin the the most important definition.

Theorem-Definition 1. A *polytope* in $\mathbb{R}^n$ is the convex hull of finitely many points *or*, equivalently, an intersection of finitely many closed half-spaces which is bounded. These are called the *V-description* and the *H-description* of the polytope respectively. A *face* of a polytope is a set which is minimized by some linear function. A *facet* is an maximal dimensional proper (i.e., not the whole polytope) face.

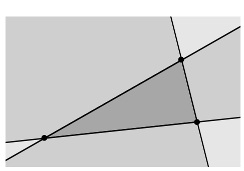


Figure 1: H-description

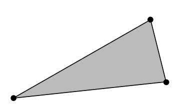


Figure 2: V-description

We rely on intuition in dimensions 2 and 3 (See figures 1 and 2) for the equivalence of these two notions of a polytope, and a proof for the general case can be found in [Zie95].

One tool for capturing some important combinatorial data about a polytope is the *f-vector*. Given a polytope P, let f_i be the number of facets of dimension i in P. Then the f-vector is $(f_0, \dots, f_d)$. For example, the f-vector of the 3 cube is

$$f = (8, 12, 6).$$

Exercise 1. Compute the f-vectors of $\Delta_4, \Delta_3 \times \Delta_1$, and $\Delta_2 \times \Delta_2$, where $\Delta_d \subset \mathbb{R}^d$ is the convex hull of $\vec{0}, e_1, \dots, e_d$, and the cartesian product of two polytopes is their cartesian product as sets (inside the product of the spaces they each lived in). *Hint*: If f is a face of P_1 of dimension d_f and e is a face of P_2 of dimension d_e, then $f \times e$ is a face of $P_1 \times P_2$ of dimension $d_e + d_f$.

Answer: $f(\Delta_4) = (5, 10, 10, 5)$, $f(\Delta_3 \times \Delta_1) = (8, 16, 14, 6)$, and $f(\Delta_2 \times \Delta_2) = (9, 18, 15, 6)$.

Definition 1. A *polytopal complex* $\mathcal{C}$ is a finite collection of polyhedra satisfying

1. the empty polyhedron is in $\mathcal{C}$,
2. if $P \in \mathcal{C}$, then all faces of P are in $\mathcal{C}$,
3. the intersection $P \cap Q$ of two polyhedra $P, Q \in \mathcal{C}$ is also a face of both P and Q.

We use the notation $|\mathcal{C}|$ to denote the underlying set of the complex $\mathcal{C}$. See figure 3 for an example.

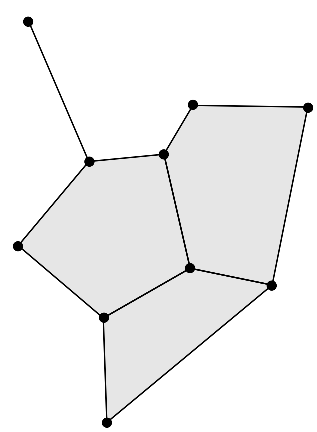


Figure 3: A polytopal complex

Given a polytopal complex $\mathcal{C}$, we can capture its combinatorial structure by considering its *face poset*, which is the finite set of polytopes in $\mathcal{C}$ ordered by inclusion. We say that two polytopal complexes are *combinatorially equivalent* if their face posets are isomorphic as posets.

Example Let P be a polytope. A *subdivision* of P is any polytopal complex $\mathcal{C}$ with the underlying space $|\mathcal{C}| = P$. The subdivision is a *triangulation* if all of the polytopes in $\mathcal{C}$ are simplices. We usually only discuss subdivisions where the 0-dimensional polytopes of the subdivision are the same as the vertices of P. Those interested to learn more about triangulations will enjoy the 500 page book *Triangulations* [DLRS10].

Exercise 2. What is the smallest number of triangles we may use to triangulate the 2-cube? What about the 3-cube?

In particular, we like to look at *regular* subdivisions. A subdivision of $P \subset \mathbb{R}^n$ is called *regular* if it arises from a polytope P' in $\mathbb{R}^{n+1}$ in the following way. If $\mathcal{C}$ consists of all *lower faces* of P', i.e., those faces you would see if you "laid on the floor" [what does this mean in more dimensions? If you take a point p in P' and draw a line through it parallel to e_{n+1}, there are no points in P' lying below p]. A common way to obtain a regular subdivision is to take a weight vector $w \in \mathbb{R}^k_{\geq 0}$ where k is the number of vertices in P, and make the polytope P' by adjoining the jth entry of w to the last entry of the jth vertex of P. See figure 4 for an example.

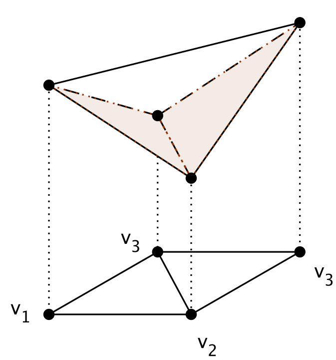


Figure 4: A regular subdivision of a parallelogram with $w = (2, 1, 2, 1)$

Exercise 3. Show that all subdivisions of an n-gon P which have the same vertices as P are regular.

Exercise 4. Give an example of a subdivision which is not regular.

Schlegel Diagrams

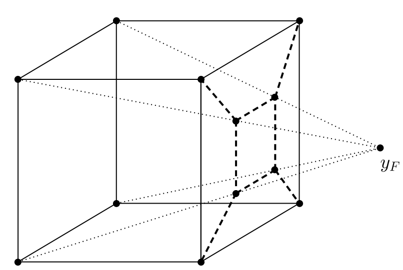
Schlegel diagrams are a good tool for understanding the combinatorial structure of a polytope. To start with, pick a facet F of the polytope, and choose a point of view y_F beyond facet F. There is a precise definition for the right notion of "beyond" is, but for now intuition will suffice: we want to pick a point outside the polytope, but so close to the facet F that if we stand at y_F and look at the polytope, all we can see is F.

Definition 2. Let P be a d-polytope in $\mathbb{R}^d$ and let F be a facet of P. Let H_F be the hyperplane spanned by F, and let y_F be as above. For each point x in the polytope, let $p(x)$ denote the point on H_F which lies on the line joining x to y_F. Then the *Schlegel diagram of* P *based at the facet* F is the image under p of all proper faces of P other than F.

Exercise 5. Show that a Schlegel diagram is a polytopal subdivision of F.

This is an important notion because the Schlegel diagram is combinatorially equivalent to the complex $C(\delta P \backslash \{F\})$, the complex of all proper faces of P other than F (a proof can be found in [Zie95]). Furthermore, it gives us a way to view a d dimensional polytope in $d-1$ dimensions. This is especially useful in the case $d = 4$.

Example We will start with $d = 3$. Here is the Schlegel diagram of the cube:



Exercise 6. Give an example of a subdivision which is regular but not Schlegel.

Can we characterize which subdivisions are Schlegel diagrams? In the case $d = 3$, there is a nice answer. Let $P \subset \mathbb{R}^3$ be a polytope, F a facet of P, and let S be the Schlegel diagram. Define a graph G by having the vertices as the 0 dimensional cells of S, and an edge between two vertices when they are each contained in the same 1 dimensional cell of S. Call G the *graph of* P. Then we have the following theorem.

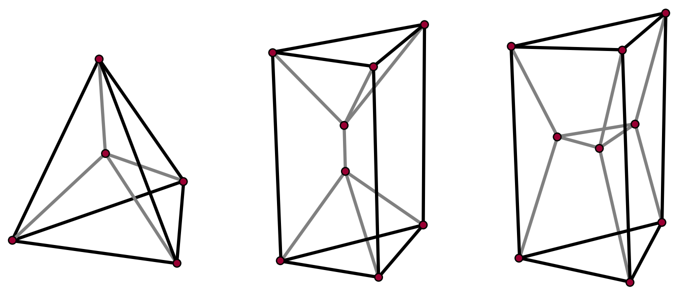
Theorem 1 (Steinitz). *The graph G is the graph of a 3-dimensional polytope if and only if it is simple, planar, and 3-connected (the removal of any 2 vertices leaves the graph connected).*

The proof of the forward direction is mostly easy: G is certainly simple, and it is planar because it came from a Schlegel diagram. It is 3-connected because of the following result:

Theorem 2 (Balinski)**.** *The graph G of a polytope $P \subset \mathbb{R}^d$ is d-connected.*

For the reverse direction, no simple proof is known. Most proofs work by arguing that each 3-connected planar graph can be built out of copies of K_4 by operations which preserve realizability.

Exercise 7. Draw Schlegel diagrams for $\Delta_4, \Delta_3 \times \Delta_1$, and $\Delta_2 \times \Delta_2$.
Answer:



Δ_4 $\quad$ $\Delta_3 \times \Delta_1$ $\quad$ $\Delta_2 \times \Delta_2$

Figure 5: Schlegel diagrams of products of simplices

References

[Chv83] Vasek Chvatal. *Linear programming.* A Series of books in the mathematical sciences. Freeman, New York (N. Y.), 1983. Reimpressions : 1999, 2000, 2002.

[DLRS10] Jesús A. De Loera, Jörg Rambau, and Francisco Santos. *Triangulations*, volume 25 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, 2010. Structures for algorithms and applications.

[Zie95] Günter M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.