Packing Polynomials on Sectors of $\mathbb{R}^2$

Madeline Brandt

Reed College, Portland, OR
mbrandt@reed.edu

January 11, 2015

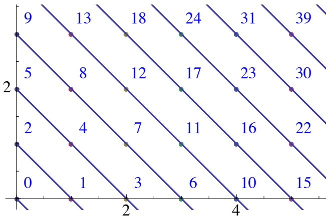
Introduction

Let $I \subset \mathbb{Z}^2$. A **packing polynomial** on I is a polynomial $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f\,|_I$ is a bijection from I to $\mathbb{N}$.

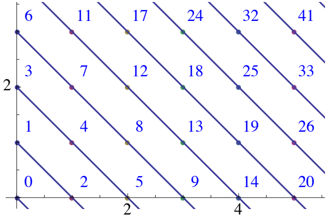
Introduction

Let $I \subset \mathbb{Z}^2$. A **packing polynomial** on I is a polynomial $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f\,|_I$ is a bijection from I to $\mathbb{N}$.

The **Cantor Polynomials**:



$$f(x, y) = \frac{(x+y)^2}{2} + \frac{x+3y}{2},$$



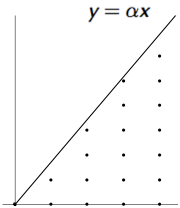
$$g(x, y) = \frac{(x+y)^2}{2} + \frac{3x+y}{2}.$$

Fueter and Pólya proved that these are the only quadratic packing polynomials on $\mathbb{N}^2$.

Sectors

For all $\alpha \in \mathbb{R}_{\geq 0}$, let

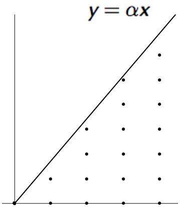
$I(\alpha) =$



$I(\alpha)$ is called a **sector**.

Sectors

For all $\alpha \in \mathbb{R}_{\geq 0}$, let

$I(\alpha) =$ 

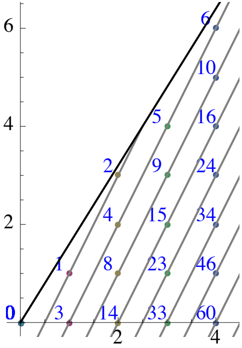
$I(\alpha)$ is called a **sector**.

- $\alpha \in \mathbb{N}$: Solved by Stanton.
- $\alpha \notin \mathbb{Q}$: Nathanson conjectured that there are no packing polynomials on $I(\alpha)$.
- $\alpha \in \mathbb{Q}$: we solved.

Example

Example:
This is a packing polynomial on $I(8/5)$, and

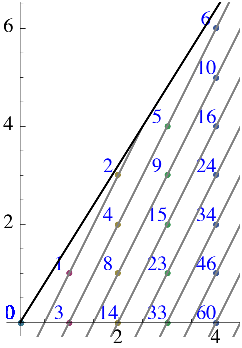
$$p(x, y) = 4\left(x - \frac{y}{2}\right)^2 - x + y.$$



Example

Example:
This is a packing polynomial on $I(8/5)$, and

$$p(x,y) = 4\left(x - \frac{y}{2}\right)^2 - x + y.$$



Definition: Let p be a quadratic packing polynomial on $I(\frac{n}{m})$. Then p is a k-**stair** polynomial if for any two consecutive integral points r, s along a line with slope $\frac{n}{m-1}$, we have $p(r) - p(s) = \pm k$.

Previous Result

Theorem (Stanton)

Let $n/m \geq 1$, and $(n, m) = 1$. If $I(n/m)$ has a quadratic packing polynomial p, then $n|(m-1)^2$ and

$$p(x, y) = \frac{n}{2}\left(x - \frac{m-1}{n}y\right)^2 + \text{ linear terms.}$$

Previous Result

Theorem (Stanton)

Let $n/m \geq 1$, and $(n, m) = 1$. If $I(n/m)$ has a quadratic packing polynomial p, then $n|(m-1)^2$ and

$$p(x, y) = \frac{n}{2}\left(x - \frac{m-1}{n}y\right)^2 + \text{ linear terms.}$$

This implies that all packing polynomials on sectors $I(n/m)$ are k-stair polynomials for some k.

Equivalence

We will say that two packing polynomials p on $I(\alpha)$ and q on $I(\beta)$ are **equivalent** if there exists a linear bijection T from $I(\alpha)$ to $I(\beta)$ such that

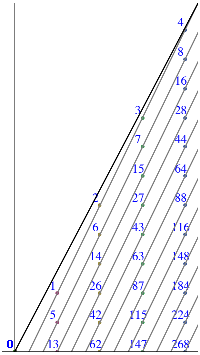
$$p = q \circ T.$$

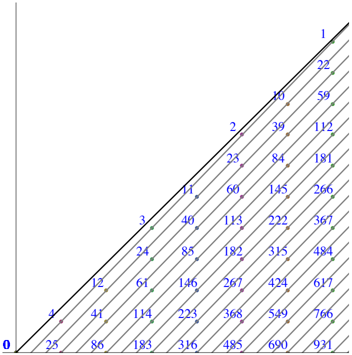
Equivalence

We will say that two packing polynomials p on $I(\alpha)$ and q on $I(\beta)$ are **equivalent** if there exists a linear bijection T from $I(\alpha)$ to $I(\beta)$ such that

$$p = q \circ T.$$

k-stair to $-k$-stair



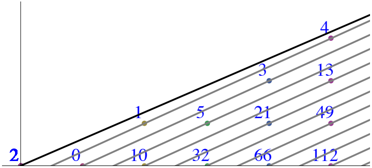


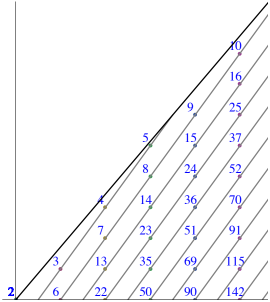
Equivalence

We will say that two packing polynomials p on $I(\alpha)$ and q on $I(\beta)$ are **equivalent** if there exists a linear bijection T from $I(\alpha)$ to $I(\beta)$ such that

$$p = q \circ T.$$

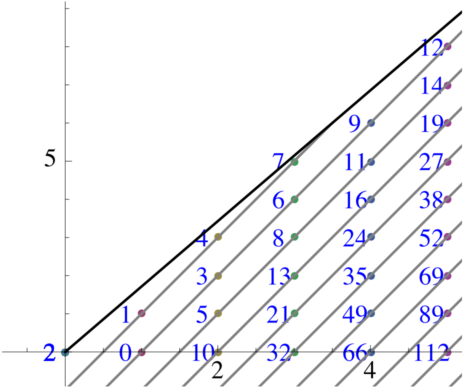
$n/m < 1$ to $n/m \geq 1$:





Properties of k-Stair Polynomials

A 3-stair packing polynomial on $I(12/7)$:



Main Result: Necessary Form

Let $l = \gcd(n, m-1)$.

Theorem (Brandt)

Let p be a k-stair packing polynomial on $I(n/m)$, where $m \neq 1$. Then (up to equivalence) $k \equiv \frac{m-1}{l} \mod \frac{n}{l}$, and

$$p(x,y) = \frac{n}{2}\left(x - \frac{m-1}{n}y\right)^2 + \left(1 - \frac{kl}{2}\right)x + \frac{2(1-m)+kl(m+1)}{2n}y + c.$$

The expression for $p(x,y)$ only depends on n, m, and k.

Main Result: Sufficient Conditions

Theorem (Brandt)

The following results give the k-stair packing polynomials on sectors $I(\frac{n}{m})$ for all k (up to equivalence).

Main Result: Sufficient Conditions

Theorem (Brandt)

The following results give the k-stair packing polynomials on sectors $I(\frac{n}{m})$ for all k (up to equivalence).

1. *1-stair polynomials:* $n \mid (m-1)^2$ *and either* $m = 1$ *or* $m - 1 \mid n$.

Main Result: Sufficient Conditions

Theorem (Brandt)

The following results give the k-stair packing polynomials on sectors $I(\frac{n}{m})$ for all k (up to equivalence).

1. *1-stair polynomials:* $n \mid (m-1)^2$ *and either* $m = 1$ *or* $m - 1 \mid n$.
2. *2-stair polynomials:* $m \equiv 9 \mod 16$ *and* $n = \frac{1}{16}(m-1)^2$.

Main Result: Sufficient Conditions

Theorem (Brandt)

The following results give the k-stair packing polynomials on sectors $I(\frac{n}{m})$ for all k (up to equivalence).

1. *1-stair polynomials:* $n \mid (m-1)^2$ *and either* $m = 1$ *or* $m - 1 \mid n$.
2. *2-stair polynomials:* $m \equiv 9 \mod 16$ *and* $n = \frac{1}{16}(m-1)^2$.
3. *3-stair polynomials:* $m \equiv 10 \mod 27$ *or* $m \equiv 19 \mod 27$ *and* $n = \frac{1}{27}(m-1)^2$.

Main Result: Sufficient Conditions

Theorem (Brandt)

The following results give the k-stair packing polynomials on sectors $I(\frac{n}{m})$ for all k (up to equivalence).

1. *1-stair polynomials:* $n \mid (m-1)^2$ *and either* $m = 1$ *or* $m - 1 \mid n$.
2. *2-stair polynomials:* $m \equiv 9 \mod 16$ *and* $n = \frac{1}{16}(m-1)^2$.
3. *3-stair polynomials:* $m \equiv 10 \mod 27$ *or* $m \equiv 19 \mod 27$ *and* $n = \frac{1}{27}(m-1)^2$.
4. *There are no k-stair packing polynomials for* $k \geq 4$.

Future Directions

1. Prove that there are no packing polynomials of degree greater than 2 on sectors of $\mathbb{R}^2$.
   - Fueter and Pólya conjectured that this was true on $\mathbb{N}^2$.
   - Lew and Rosenberg have proved that there are no degree 3 or 4 packing polynomials on $\mathbb{N}^2$.

Future Directions

1. Prove that there are no packing polynomials of degree greater than 2 on sectors of $\mathbb{R}^2$.
   - Fueter and Pólya conjectured that this was true on $\mathbb{N}^2$.
   - Lew and Rosenberg have proved that there are no degree 3 or 4 packing polynomials on $\mathbb{N}^2$.
2. Prove that there are no packing polynomials on irrational sectors. (Nathanson)

Acknowledgments

- Joseph Gallian
- The University of Minnesota Duluth Research Experience for Undergraduates in Mathematics Program, including the advisors and visitors to the program.
- This research was supported by the National Science Foundation (grant number DSF 1358659) and the National Security Agency (NSA grant H98230-13-1-0273).
- This presentation was supported by a Reed College Undergraduate Research Opportunity Grant.