Tropical Geometry of Curves

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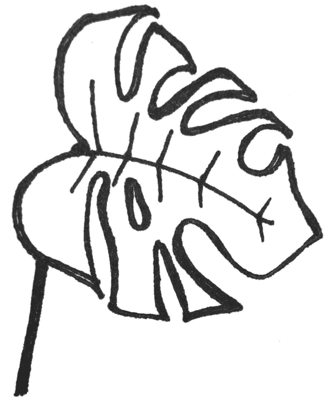


1 Introduction

Tropical geometry is a new subject which creates a bridge between the two islands of algebraic geometry and combinatorics. It has many fascinating connections to other areas as well. The aim of this lecture is to introduce tropical geometry and to provide a glimpse of a research area in tropical geometry, namely, computing abstract tropicalizations of curves.

2 Background

Definition 2.1. The **Puiseux series** K is the field of formal power series with rational exponents and nonzero coefficients in $\mathbb{C}$:

$$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$$

for a_i an increasing sequence of rational numbers which have a common denominator.

The Puiseux series are equipped with a special function $\nu : K \to \mathbb{R}$, called the **valuation**. The valuation is given by taking $\nu(c) = a_1$. The Puiseaux series are algebraically closed.

Tropicalization is a process we apply to varieties over a field with a valuation. The Puiseux series are not the only field with a valuation, but to simplify matters we only use this example in this talk.

We denote by R the set of all elements with nonnegative valuation. It is a local ring with maximal ideal $\mathfrak{m}$ given by all elements with positive valuation. The quotient ring is denoted by $\Bbbk$ and it is called the **residue field**.

3 Hypersurfaces

Hypersurfaces are the most natural starting place for studying tropical geometry. Consider the ring $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ of Laurent polynomials over K.

Definition 3.1. Given a Laurent polynomial

$$f = \sum_{u \in \mathbb{Z}^n} c_u x^u,$$

we define its **tropicalization** to be the real valued function on $\mathbb{R}^n$ that is obtained by replacing each c_u by its valuation and preforming all additions and multiplications in the **tropical semiring** $(\mathbb{R}, \oplus, \otimes)$:

$$\mathrm{trop}(f)(w) = \min_{u \in \mathbb{Z}^n} (\mathrm{val}(c_u) + u \cdot w) .$$

Classically, the variety of the Laurent polynomial f is a hypersurface in the algebraic torus $T^n = (K^*)^n$. We now define the tropical hypersurface associated to f.

Definition 3.2. The **tropical hypersurface** $\mathrm{trop}(V(f))$ is the set

$$\{w \in \mathbb{R}^n \mid \text{the minimum in } \mathrm{trop}(f)(w) \text{ is achieved at least twice}\}$$

Theorem 3.3 (Kapranov)**.** *The set trop$(V(f))$ is the same as*

$$\overline{\{(\nu(y_1), \dots, \nu(y_n)) \mid (y_1, \dots, y_n) \in V(f)\}}$$

Example 3.4. Here we compute the **tropical line**, which is the classic first example. Let $f = x + y + 1$ in the field $\mathbb{C}\{\{t\}\}$. Then,

$$\begin{aligned} \mathrm{trop}(f)(w) &= \min (0 + (1,0) \cdot w, 0 + (0,1) \cdot w, 0) , \\ &= \min(w_1, w_2, 0). \end{aligned}$$

So, where is this minimum achieved twice? We can break this down in to 3 cases.

1. w_1 and 0 are the winners: This happens when $w_1 = 0$ and $w_2 \geq 0$. So, this is the ray $\mathrm{pos}(e_2)$.

2. w_2 and 0 are the winners: This happens when $w_2 = 0$ and $w_1 \geq 0$. So, this is the ray $\mathrm{pos}(e_1)$.

3. w_1 and w_2 are the winners: This adds to our tropical variety the ray $\mathrm{pos}(-1, -1)$.

So, the tropical variety is pictured in Figure 1

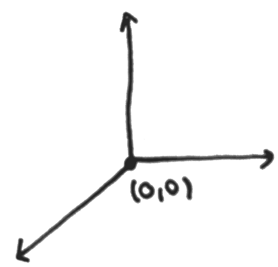


Figure 1: The tropical line studied in Example 3.4.

In practice, when you wish to compute a tropical hypersurface, there is a very practical method (this is especially good in two dimensions).

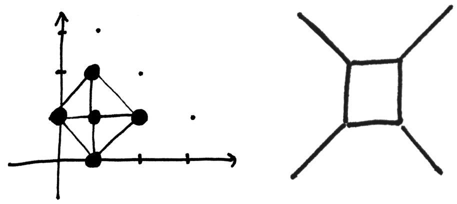
Proposition 3.5. *Let* $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ *be a Laurent polynomial. The tropical hypersurface trop*$(V(f))$ *is the* $(n-1)$*-skeleton of the polyhedral complex dual to the regular subdivision of the newton polytope of* f *induced by the weigths val*(c_u) *on the lattice points in Newt*(f)*.*

I didn't define most of the terms in this proposition, but hopefully from the next example it will make sense how to use it.

Example 3.6. Consider the plane curve over the Puiseux series in t defined by the equation

$$f(x, y, z) = txz^2 + tyz^2 + xyz + txy^2 + tx^2y.$$

To find the embedded tropical hypersurface corresponding to this curve, we do the following. Make the Newton polygon, which is the convex hull of the exponent vectors. Then, find the regular subdivision induced from the weights given by the valuations of the coefficients. Then, the tropical curve will be dual to this (and rotated 180 degrees).

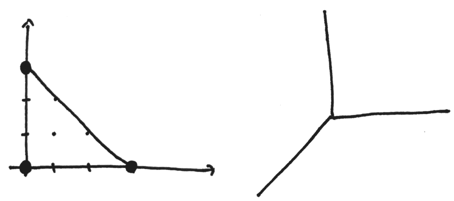


At this point, we have found the embedded tropicalization of $V(f)$.

Now, we repeat this computation, but first we will do a change of coordinates. Consider

$$\begin{aligned} f(x-y-z, y-x-z, z+x+y) = & -x^3 - y^3 + z^3 \\ & + \text{more middle terms} \\ & + \text{terms higher order in } t. \end{aligned}$$

Then, we can compute the embedded tropicalization again.



We notice now that we have two very different answers, even though we were tropicalizing the same curve. Perhaps, this should be disturbing– shouldn't there be a way to associate a "tropical curve" to each isomorphism class of algebraic curve?

4 Abstract Tropicalization

Now, our goal is to define an object called the **abstract tropicalization** which can be associated to every curve, and does not depend on the embedding.

Let C be a smooth curve over K. Given some equations defining the curve C, the coefficients of these equations will possibly involve the uniformizer t. Informally, we can think of the uniformizer t as "going to zero," and when $t = 0$, we will see some special and possibly singular behavior. Generically, the curve will be smooth, but it could limit to something singular. More formally, we need a **model** of the curve.

Definition 4.1. We say a **model** $\mathcal{C}$ for a curve C over K is a flat and finite type scheme over R whose generic fiber is isomorphic to C. We call this model **semistable** if the special fiber

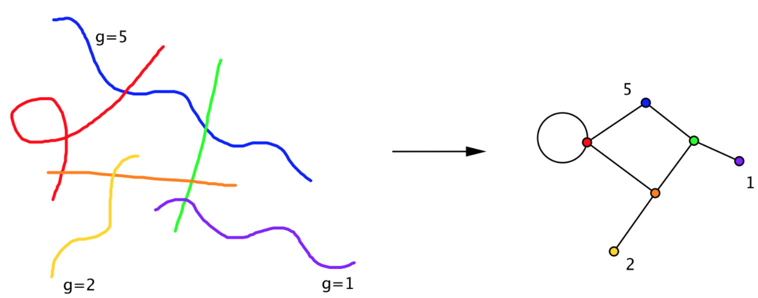
$$\mathcal{C}_k = C \times_R k$$

is a semistable curve over k, meaning:

1. It is reduced, connected, and only has nodal singularites;
2. every rational component has at least 2 singular points.

Definition 4.2. The **dual graph** of $\mathcal{C}_k$ has vertices corresponding to the irreducible components of $\mathcal{C}_k$, and edges corresponding to nodes.

Here is an example of a schematic of the special fiber of a curve C and on the right, its dual graph.



To define the **abstract tropicalization** of the curve C, we add a bit of extra data.

1. **Vertex Weights**: we add weights to the vertices by assigning to each vertex the **genus** of the corresponding component.
2. **Edge Lengths**: We add edge lengths in the following way. Given an edge corresponding to a node q between two components C_i and C_j, the completion of the local ring $\mathcal{O}_{\mathcal{C},q}$ is isomorphic to $R[[x,y]]/(xy - f)$ where $\nu(f) > 0$. Then, we define the length of the edge e_{ij} to be $\nu(f)$.

So, why is this hard? Depending upon the equations that the curve arrives to you with, the singularities when $t = 0$ could be worse than nodes. By the **semistable reduction theorem**, we are always guaranteed in the abstract that we can replace a bad special fiber with a good one. However, this process is not algorithmic, and this proves to be the main difficulty in finding the abstract tropicalization of a curve in explicit examples.

The problem of computing the abstract tropicalization has been studied in several classes of curves.

1. In genus 1, the answer has been known for some time; one simply takes the valuation of the j-invariant of the curve and if it is negative, then the abstract tropicalization will be a cycle of length negative of this valuation.
2. The problem of computing the Berkovich skeleton for genus 2 curves was done systematically by studying the ramification data in [RSS14] and using Igusa invariants in [Hel16].
3. In the case of hyperelliptic curves, this problem was studied in [Cha13] and later solved in [BBCar] using ramification data and admissible covers. In [Hel17], Helminck presents criteria to reconstruct Berkovich skeleta using Laplacians on metric graphs.
4. We [Brandt-Helminck] have also applied these techniques to the superelliptic case, to obtain an algorithm for tropicalizing superelliptic curves.

References

[BBCar] Barbara Bolognese, Madeline Brandt, and Lynn Chua, **From curves to tropical jacobians and back**, Combinatorial Algebraic Geometry (G.G. Smith and B. Sturmfels, eds.), to appear.

[Cha13] Melody Chan, **Tropical hyperelliptic curves**, Journal of Algebraic Combinatorics. An International Journal **37** (2013), no. 2, 331–359.

[Hel16] Paul Alexander Helminck, **Tropical Igusa invariants and torsion embeddings**, arXiv:1604.03987 (2016).

[Hel17] ______, **Tropicalizing abelian covers of algebraic curves**, arXiv:1703.03067 (2017).

[RSS14] Qingchun Ren, Steven V. Sam, and Bernd Sturmfels, **Tropicalization of classical moduli spaces**, Mathematics in Computer Science **8** (2014), no. 2, 119–145.