Computing Free Resolutions in Macaulay2

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October 6, 2015

Introduction

We will let $R = k[x_1, \ldots, x_r]$.

Definition 1. A *free resolution* of an R-module M is a complex

$$\mathcal{F} : \cdots \to F_n \xrightarrow{\phi_n} \cdots \to F_1 \xrightarrow{\phi_1} F_0$$

of free R modules such that $\operatorname{coker}(\phi_1) = F_0/\operatorname{im}(\phi_1) = M$ and such that $\mathcal{F}$ is exact. The image of the map ϕ_i is called the *i-th syzygy module of M*.

A free resolution is *minimal* if the image of each of the maps ϕ_i is contained in mF_{i-1}, where $m = (x_1, \ldots, x_n)$, meaning the corresponding matrices contain no entries from the field k. Informally, we may think of a free resolution being minimal if each of the matrices corresponding to these maps has no entries in the field k. Otherwise, we would have a relation among the bases and so some of them were not strictly necessary.

Example Let $I = (x^2, y^3, z^6) \subset \mathbb{Q}[x, y, z] = R$. To resolve R/I, we must first find the kernel of the map $\phi_1 : R^3 \to R$ is given by the matrix $[x^2 \ y^3 \ z^6]$ whose image is I. In this case, we can do this by eyeballing, and so we get $\phi_2 : R^3 \to R^3$ is given by

$$\begin{bmatrix} y^3 & 0 & z^6 \\ -x^2 & z^6 & 0 \\ 0 & -y^3 & -x^2 \end{bmatrix}.$$

Then, we may compute the kernel of the above map, to find $\phi_3 : R \to R^3$ is $[z^6 \ -x^2 \ y^3]^T$. Then we have successfully computed this free resolution by repeatedly computing kernels of maps.

A Computer Experement

In Macaluay2 we can compute free resolutions of R/I:

```
R = QQ[x,y,z]
I = ideal(x^2,y^3,z^6)
rs = res I
rs.dd
```

This will show you the resolution we computed by hand. How does Macaulay2 compute free resolutions? A first guess is that it does this in the same way we did it: by repeatedly computing kernels. Here is some code that computes free resolutions in this way:

```
badRes = I -> (
f := gens I;
L:= {};
while f != 0 do
(L = append(L,f);
f =gens ker gens image f);
append(L,f);
chainComplex(L))
```

You can load this command into Macaulay2 by dowloading the file badres.M2, navigating yourself to the appropriate directory, and typing

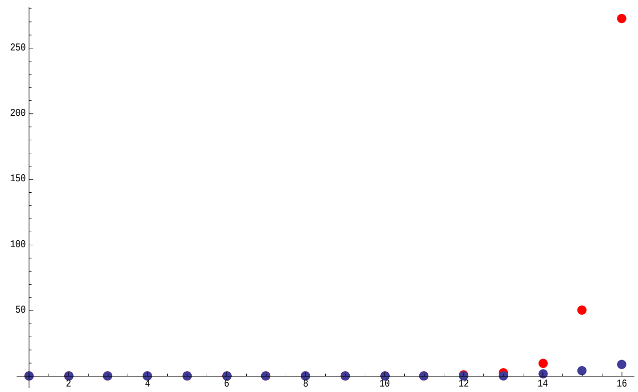
```
load "badres.M2"
```

Then, we will compare the output of Macaulay2's res command and my badRes command. Notice that the generators are the same, but out of order. This is the first indication that something different is going on.

Now, compute

```
clearAll
R = QQ[x_0 ..  x_14]
load "badres.M2"
I = ideal vars R
time badRes I
time res I
```

We can see that badRes takes much more time. I have compared the times below in the following plot:



where the x-axis is n the number of variables, and the y-axis is time used to compute the free resolution of $I = (x_1, \ldots, x_n)$. Blue dots are for res and red dots are for badRes. We can see that badRes computes much more slowly than res.

This indicates Macaulay2 is computing free resolutions differently than the way we first guessed.

Free Resolutions for Free

The efficient algorithm used by Macaulay2 for computing free resolutions is originally do to Schreyer [Sch91].

Let F be a free R module with basis and let M be a submodule of F. Fix a monomial order for F. Let $G = \{g_1, \ldots, g_t\}$ be generators of M. Let

$$m_{ij} = \frac{\text{in}(g_i)}{\gcd(\text{in}(g_i), \text{in}(g_j))},$$

and

$$S(g_i, g_j) = m_{ji} g_i - m_{ij} g_j,$$

for all pairs i, j for which g_i and g_j involve the same basis element of F.

Buchberger's Algorithm tells us that when we compute $S(g_i, g_j)$ and the remainder r_{ij} of this S-polynomial upon division by G, then if r_{ij} is nonzero, we add this to G and continue pairwise testing the members of G. Buchberger's criterion says that we are done when $r_{ij} = 0$ for all i and j in $\{1, \ldots, t\}$. So, for each i and j, when we compute a Gröbner basis, we have already computed something of the form

$$S(g_i, g_j) = \sum_u f_u^{(ij)} g_u + r_{ij}.$$

If G is a Gröbner basis, then by Buchberger's criterion, $r_{ij} = 0$. Then we have,

$$0 = \sum_u f_u^{(ij)} g_u - m_{ji} g_i + m_{ij} g_j.$$

This is a syzygy on the g_i.

Let $\bigoplus R\epsilon_i$ be a free module with basis ϵ_i corresponding to the g_i. Let

$$\phi : \bigoplus R\epsilon_i \to F, \qquad \epsilon_i \mapsto g_i.$$

Let $\tau_{ij} = \sum_u f_u^{(ij)}\epsilon_u - m_{ji}\epsilon_i + m_{ij}\epsilon_j$.

Theorem 1 (Schreyer 15.10 [Eis95]). *The τ_{ij} defined above generate the syzygies on the g_i. In a monomial ideal, the syzygies are generated by the S-polynomials.*

Additionally, the τ_{ij} are a Gröbner basis for the syzygies with respect to the order $>$, where $>$ is the monomial order on $\bigoplus R\epsilon_j$ defined by taking $m\epsilon_u > n\epsilon_v$ iff

$$in(mg_u) > in(ng_v) \quad \text{with respect to the order on } F, \text{ or}$$

$$in(mg_u) = in(ng_v) \quad \text{but} \;\; u < v.$$

Under this ordering, we also have $in(\tau_{ij}) = m_{ji}\epsilon_i$.

So, given a module M, to compute a free resolution we do the following. First compute a Gröbner basis for M, and using Buchberger's criteria as above, compute the syzygies on the Gröbner basis elements. To obtain the syzygies on the original generators of M, substitute into these syzygies the expressions for the Gröbner basis elements in terms of the original generators.

Example Let $g_1 = x^2$ and $g_2 = xy + y^2$. Use the lex order where $x > y$. Last week, Liz showed us that we obtain a Gröbner basis if we add in the polynomial

$$g_3 = y^3 = yg_1 - xg_2 + yg_2.$$

Now, we test for Buchberger's criterion, and find that

$$\begin{aligned} r_{12} &= 0 = yg_1 - xg_2 + yg_2 - g_3, \\ r_{23} &= 0 = y^2g_2 - xg_3 - yg_3, \\ r_{13} &= 0 = y^3g_1 - x^2g_3. \end{aligned}$$

Then set

$$\begin{aligned} \tau_{12} &= y\epsilon_1 - x\epsilon_2 + y\epsilon_2 - \epsilon_3, \\ \tau_{23} &= y^2\epsilon_2 - x\epsilon_3 - y\epsilon_3, \\ \tau_{13} &= y^3\epsilon_1 - x^2\epsilon_3. \end{aligned}$$

.

So far, we know that g_1, g_2, g_3 form a Gröbner basis and that $\tau_{12}, \tau_{23}, \tau_{13}$ generate the syzygies on them.

To obtain the syzygies on the original generators, we recall that $g_3 = yg_2 + (y-x)g_2$. So, we set $\tau_{12} = 0$, and substitute $\epsilon_3 = y\epsilon_1 - x\epsilon_2 + y\epsilon_2$ into the expressions for τ_{23}, τ_{13}. Then we find that

$$\tau_{23} = (-xy - y^2)\epsilon_1 + x^2\epsilon_2, \qquad \tau_{13} = (y^3 - x^2y)\epsilon_1 + (-x^2y + x^3)\epsilon_2.$$

However, this is not minimal since $\tau_{13} = (-y+x)\tau_{23}$, and so the syzygies on the original generators are completely generated by τ_{23}.

This algorithm is what Macaulay2 uses to compute free resolutions, and this is much more efficient than repeatedly computing kernels. Furthermore, it can give us the entire free resolution, since the τ_{ij} give a Gröbner basis for the first Syzygy module.

Hilbert Syzygy Theorem

We are now ready to prove Hilbert's Syzygy Theorem, so we will restate it below.

Theorem 2 (Hilbert Syzygy Theorem 15.11 [Eis95])**.** *Suppose that the g_i are arranged such that whenever in(g_i) and in(g_j) involve the same basis element ϵ of F, say in$(g_i) = n_i\epsilon$ and in$(g_j) = n_j\epsilon$ with $n_i, n_j \in R$, we have $i < j$ implies $n_i > n_j$ in the lexicographic order. If the variables $x_1, \ldots, x_s$ are missing from the initial terms of the g_i, then*

1. *The variables $x_1, \ldots, x_{s+1}$ are missing from the in(τ_{ij})*
2. *$F/(g_1, \ldots, g_t)$ has a free resolution of length less than or equal to $r - s$.*

In particular, every finitely generated R-module has a free resolution of length less than or equal to r.

By the previous theorem, we have that $\text{in}(\tau_{ij}) = m_{ji}\epsilon_i$. Since $\text{in}(g_i) \geq \text{in}(g_j)$ we have that $m_i \geq m_j$ in the lex order. Then x_{s+1} appears in m_i with at least as high a power as in m_j, so it does not appear in m_{ji}. This proves (1).

For (2), we proceed by induction. First, suppose $r - s = 0$. Then none of the x_i appear in the $\text{in}(g_i)$, so then $\text{in}(g_1, \ldots, g_t)$ is the free submodule of F generated by the ϵ_i which appear in the $\text{in}(g_i)$. Let F' be the free module generated by the other ϵ_j, and consider

$$F' \subset F \to F/(g_1, \ldots, g_t).$$

Since $F/(g_1, \ldots, g_t)$ has a basis of monomials in F' (Theorem 15.3: $\{\text{monomials}\}\backslash\text{in}(M)$ generates F/M) this is actually an isomorphism, so $F/(g_1, \ldots, g_t)$ is free.

Now suppose $r - s > 0$. By (1), we have that $x_1, \ldots, x_{s+1}$ are missing from the τ_{ij}. We may order the τ_{ij} so that they satisfy the same hypotheses as those on the g_i. Then by induction,

$$\bigoplus R\epsilon_i/(\{\tau_{ij}\})$$

has a free resolution of length $r - s - 1$. Then putting this together with the map $\phi : \bigoplus R\epsilon_i \to F$, we get the desired free resolution of $F/(g_1, \ldots, g_t)$.

Free Resolutions of Monomial Ideals: 3 Variables

Let $R = k[x, y, z]$, and let $I = (m_1, \dots, m_r)$ be a monomial ideal in R. As we saw in the previous section, the first syzygies of I are generated by the S-polynomials:

$$\sigma_{ij} = \frac{\mathrm{lcm}(m_i, m_j)}{m_i}\epsilon_i - \frac{\mathrm{lcm}(m_i, m_j)}{m_j}\epsilon_j.$$

There are $\binom{r}{2}$ of these in total, but they do not generate the syzygies minimally.

Definition 2. Define a graph associated to I called the *Buchberger graph* $Buch(I)$. The vertex set will be the numbers $\{1, \dots, r\}$ and there will be an edge $\{i, j\}$ whenever there is no m_k such that $m_k \mid \mathrm{lcm}(m_i, m_j)$ and the degree of m_k is different from $\mathrm{lcm}(m_i, m_j)$ in every variable that occurs in $\mathrm{lcm}(m_i, m_j)$.

Then we have the following proposition:

Theorem 3 ([MS99]). *The syzygies on I are generated by the σ_{ij} such that $\{i, j\}$ is an edge of $Buch(I)$.*

In fact, we can obtain the entire free resolution from this graph $Buch(I)$. Suppose $Buch(I)$ is planar, and $Buch(I)$ has v vertices, e edges, and f faces. Then

$$\mathcal{F}_{\mathcal{G}} : 0 \to R^f \xrightarrow{\delta_F} R^e \xrightarrow{\delta_E} R^v \to R \to 0$$

is a free resolution, where for an edge e_{ij},

$$\delta_E(e_{ij}) = \sigma_{ij},$$

and for a face W,

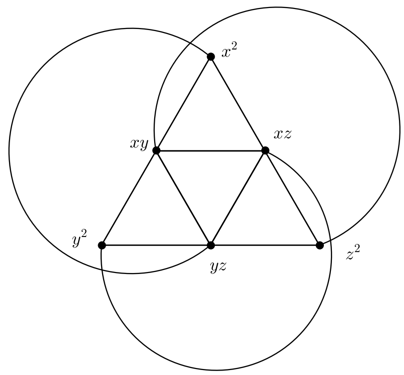
$$\delta_F(W) = \sum_{e \in W} \pm \frac{\mathrm{lcm}(v \in R)}{m_{ij}},$$

where the sign is positive when the edge is oriented clockwise around W.

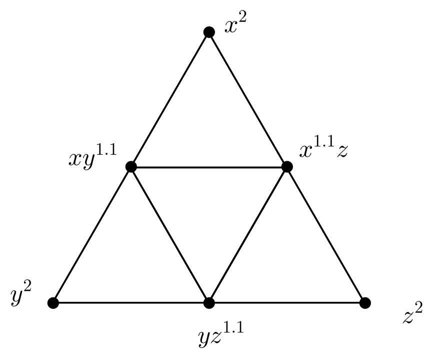
One complication is that the graph $Buch(I)$ is not necessarily planar, and will not always give a minimal free resolution. However, in the case when I is *strongly generic*, the graph $Buch(I)$ is planar, connected, and gives a minimal free resolution of I. So when I is not strongly generic, we may approximate it by a strongly generic ideal to obtain a planar graph which we may specialize to I. Then, removing extra edges, this planar graph will give a minimal free resolution of I.

Theorem 4 ([MS99]). *Every monomial ideal I in $k[x, y, z]$ has a minimal free resolution by some planar graph.*

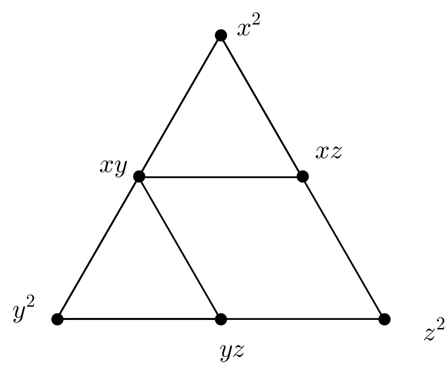
Example Let $I = (x^2, xy, y^2, yz, z^2, xz)$. Then $Buch(I)$ is



If we approximate I by the ideal $I_\epsilon = (x^2, xy^{1.1}, y^2, yz^{1.1}, z^2, x^{1.1}z) \subset k[x^{1/10}, y^{1/10}, z^{1/10}]$, then $Buch(I_\epsilon)$ is



If we consider this as a subgraph of $Buch(I)$ in the natural way, then we notice that one edge on the interior triangle will be redundant, because $y\epsilon_{xz} - z\epsilon_{xy} = (y\epsilon_{xz} - x\epsilon_{yz}) + (x\epsilon_{yz} - z\epsilon_{xy})$, so we can remove one of these edges to obtain:



Then a minimal free resolution of I is given by

$$0 \to R^3 \to R^8 \to R^6 \to R \to 0.$$

In general, for the ideal $I = (x, y, z)^m$, we can take the same triangle graph but scaled up, and for each "down" triangle, remove an edge (it will be redundant for the same reason as above). Counting the vertices, edge, and faces, a free resolution in this case is given by

$$0 \to R^{\binom{n+1}{2}} \to R^{n(n+2)} \to R^{\binom{n+2}{2}} \to R \to 0.$$

These "geometric" resolutions can be extended for higher variables.

Conclusions

We found that some syzygies are computed while computing a Gröbner basis of an ideal, and in fact, these syzygies generate all syzygies on the ideal. This observation gives an efficient method for computing free resolutions, and gives a geometric method for finding free resolutions of some monomial ideals.

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