Tropical Grassmannian and Dressian of a Matroid

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Abstract

The Grassmannian Gr_M of a realizable matroid M is an algebraic variety which provides an example of a realization space for the matroid M. In this talk, we give a brief introduction to tropical geometry, and then we study the properties of two tropical objects related to Gr_M, namely its tropicalization and a tropical prevariety called the Dressian, whose points give all regular matroid subdivisions of the matroid polytope of M. We will compute examples and study them in detail. No prior knowledge of tropical geometry is assumed.

1 Some Tropical Geometry

Let K be an algebraically closed field. A **valuation** on K is a function $v : K \to \mathbb{R} \cup \infty$ satisfying the following properties:

1. $v(a) = \infty$ if and only if $a = 0$,
2. $v(ab) = v(a) + v(b)$,
3. $v(a + b) \geq \min(v(a), v(b))$

Example. The **trivial valuation** sends all nonzero elements to 0 and 0 to ∞. The **puiseux series**

$$\mathbb{C}\{\{t\}\} = \left\{ \sum_{i=1}^{\infty} a_i t^{b_i/n} \mid a_i \in \mathbb{C}, b_i \text{ increasing integer sequence}, n \in \mathbb{N} \right\}$$

has a valuation sending an element of the above form to b_1/n.

In tropical geometry, we study varieties inside the torus $(K^*)^n$.

Definition 1. Let I be an ideal in $K[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$, where K is a field with nontrivial valuation. The **tropical variety** over K associated to I is the closure in $\mathbb{R}^{n+1}$ of

$$\operatorname{trop}(V(I)) = \{(v(y_0), \ldots, v(y_n)) \mid (y_0, \ldots, y_n) \in V(I)\}.$$

By the Structure Theorem for Tropical Varieties (Theorem 3.3.5, [MS15]), this is the support of a a balanced, weighted polyhedral complex which is pure of the same dimension as the variety. Moreover, it is connected through codimension 1.

By the Fundamental Theorem of Tropical Algebraic Geometry (Theorem 3.2.3, [MS15]), this coincides with

$$\bigcap_{f \in I} \operatorname{trop}(V(f))$$

where $\operatorname{trop}(V(f))$ is the tropical hypersurface associated to the polynomial f (See Definition 3.1.1, [MS15]).

By a theorem, for every ideal there exists a finite set B called a **tropical basis** over which it is sufficient to take the above intersection. However, it can be difficult to find such a B. In light of this, a set of the form

$$\bigcap_{f \in B'} \operatorname{trop}(V(f))$$

where B' is any collection of generators for I is called a **tropical prevariety**.

2 Grassmannians of Matroids

The **Grassmannian** (with the Plücker embedding) $G(r,m) \subset \mathbb{P}^{\binom{m}{r}-1}$ is the image of $K^{r \times m}$ under the **Plücker embedding**, which sends a matrix to the vector of its $r \times r$ minors, its **Plücker coordinates**. It is a smooth algebraic variety defined by equations called the **Plücker relations**. Points of this variety correspond to r-dimensional linear subspaces of K^m.

We now study the open subset $G^0(r,m)$ of the Grassmannian, which parametrizes subspaces whose Plucker coordinates are all nonzero, and its tropicalization. This is defined by the Plücker relations in the Laurent polynomial ring in $\binom{m}{r}$ variables. Points in this variety correspond to matrices where no Plücker coordinate (minor) vanishes. In other words, these are matrices which give the uniform matroid of rank r on $[m]$.

More generally, let M be a matroid of rank r on the set $E = [m]$. For any basis σ of M, we introduce a variable p_σ. From here, we make a Laurent polynomial ring over the field K:

$$K[p_\sigma^{\pm 1} \mid \sigma \text{ is a basis of } M].$$

Let I_M be the ideal in this ring whose generators are obtained from the Plücker relations by setting all variables not indexing a basis to zero. The Plücker relations can be obtained in `Macaulay2` by writing `Grassmannian(r-1,m-1)`.

The variety $\mathrm{Gr}_M = V(I_M)$ is the **realization space** of the matroid M, in the following sense. Points in Gr_M correspond to equivalence classes of $r \times m$ matrices that realize the matroid M. Equivalently, this is the variety of all r-dimensional linear subspaces of K^m whose nonzero Plucker coordinates are the bases of M. This variety is empty if and only if M is not realizable over K.

3 Tropical Grassmannian and Dressian

The tropicalization $trop(\mathrm{Gr}_M)$ of the realization space is called the **tropical Grassmannian of M**. We note here that when the rank is 2, the generators for I_M described above form a tropical basis. Otherwise, they almost always do not.

Definition 2. The **Dressian** Dr_M of the matroid M is the tropical prevariety obtained by intersecting the tropical hypersurfaces of the described generators for I_M.

By definition, we have that $\mathrm{trop}(Gr_M) \subseteq Dr_M$, and equality holds if and only if the quadratic Plücker relations form a tropical basis. So, when $r = 2$ we have equality, and typically when $r \geq 3$ we do not have equality.

4 Matroid subdivisions and the Dressian

The **matroid polytope** of M is the convex hull of the indicator vectors of the bases of M.

Theorem 1 (GGMS Theorem, 4.2.12 in [MS15])**.** *A polytope P with vertices in $\{0,1\}^{n+1}$ is a matroid polytope if and only if every edge of P is parallel to $e_i - e_j$.*

A subdivision of the matroid polytope P_M is a **matroid subdivision** if all of its edges are translates of $e_i - e_j$. Then, every cell of the subdivision is a matroid polytope. Every vector w in $\mathbb{R}^{|B|}/\mathbb{R}1$ induces a regular subdivision Δ_w of the polytope P_M.

Proposition 1. *Let M be a matroid, and let $w \in \mathbb{R}^{|B|}$. Then w lies in the Dressian Dr_M if and only if the corresponding regular subdivision is a matroid subdivision.*

Example. Let $M = U(2,4)$, the uniform rank 2 matroid on 4 elements (every pair is a basis).

First, we compute the tropical Grassmannian and Dressian. To do this, we need to find I_M. The Plücker relations for $Gr(2,4) \subset \mathbb{C}[p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}]$ are given by a single equation

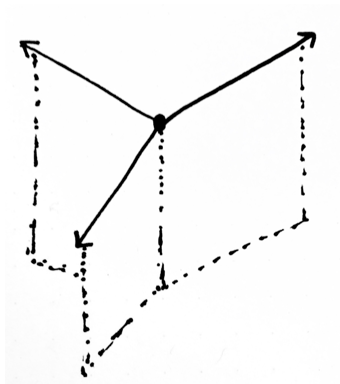
$$p_{03}p_{12} - p_{02}p_{13} + p_{01}p_{23}.$$

These were computed in `Macaulay2`. Then, the tropical Grassmannian equals the Dressian (since we are in the hypersurface case). We can compute the tropical variety in `gfan`. This tells us the following data:

The ambient dimension is 5, the dimension of the tropical variety is 5, the lineality dimension is 4, and there are 3 rays (which form the maximal cones):

$$(-2, 1, 1, 1, 1, -2)$$
$$(1, -2, 1, 1, -2, 1)$$
$$(1, 1, -2, -2, 1, 1)$$

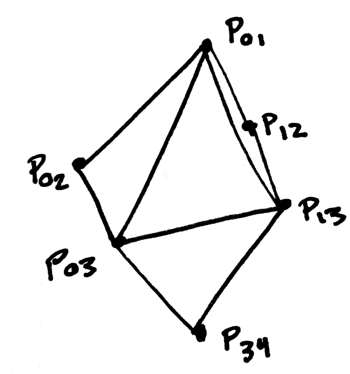
My "cartoon" for what this looks like is given below. For each plane in the picture one should imagine a 5-dimensional linear space.



Now, let us study the matroid polytope. The ground set $E = \{0, 1, 2, 3\}$ has 4 elements, and any two elements together form a basis. So, the matroid polytope is given by the hypersimplex

$$\Delta(2,4) = \mathrm{conv}((1,1,0,0),(1,0,1,0),(1,0,0,1),(0,1,1,0),(0,1,0,1),(0,0,1,1)),$$

and it looks like



We can compute in `Polymake` the regular subdivisions corresponding to points in our fan, and confirm that they are matroidal. For example, the subdivision given by the first ray has weight vector

$$w = (2, -1, -1, -1, -1, 2)$$

which subdivides the polytope in to two rectangular pyramids, the top half and the bottom half. Each of these is a matroid polytope corresponding to setting either p_{01} or p_{34} to 0 (making the corresponding pair a non-basis).

5 What else is known about tropical Grassmannians and Dressians?

One large area of study is the relationship of tropical linear spaces to tropicalized linear spaces, and the way in which these objects are parametrized by the Dressian and tropical Grassmannian respectively. We will not go into detail on this correspondence in the talk but the interested reader may look at [MS15], Chapter 4.

Here we give some open questions related to Dressians and the results in each of these directions.

1. What is the dimension of the Dressian, and what sorts of subdivisions to the maximal cones correspond to?

   In [JS17] the authords give bounds on the dimension (theorem 31).

   In [HJJS09], they show that the dimension of the dressian $Dr(3, n)$ is of order $\mathcal{O}(n^2)$. It is later proved in [JS17] that for fixed d the dimension of the Dressian is of order $\mathcal{O}(n^{d-1})$

2. Characterize rays of $Dr(3, n)$ (i.e., coarsest matroid subdivisions of $\Delta(3, n)$). [HJJS09]

   Some amount of highly detailed work exhibiting specific types of rays is given in [JS17].

3. Are all rays of $Dr(3, n)$ also rays of $Gr(3, n)$? [HJJS09] What about in general [HJS14]?

   The authors show ([JS17], Corollary 42) that this is false for $r = 4, n \geq 11$ and $r \geq 5, n \geq 10$. It is known to be true for $r = 3$ and a few small n.

4. Is it feasible to compute $Gr(3, 8)$ [HJS14]?

   In [HJJS09] they compute the tropical Dressian and Grassmannian for the case (3,7).

5. Is it feasible to compute $Dr(4, 8)$ [HJS14]?

   In [HJS14], they compute the Dressian for the $(3, 8)$ case.

6. Are there interesting examples of matroids and their Dressians?

   In [HJJS09], the authors also compute the Dressian and Grassmannian for the Pappus matroid, and give an in-depth analysis of the corresponding matroid subdivisions of the matroid polytope.

References

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