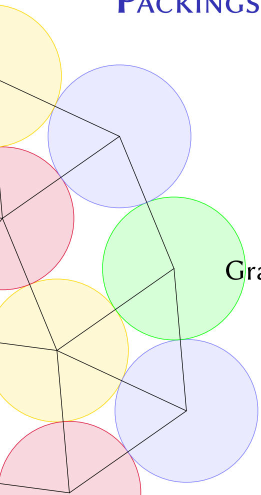
Packings of Four Equal Circles on Flat Tori



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[1]Reed College, Portland, OR
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Grand Valley State Mathematics REU, 2013

Our Goal

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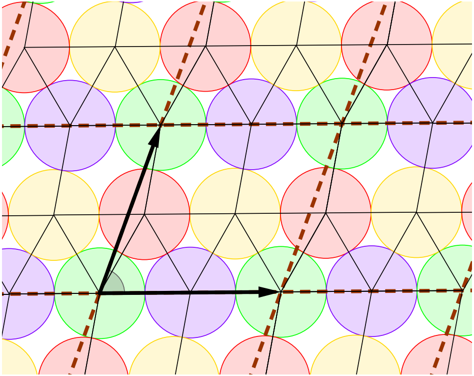

An Optimal Packing of Four Equal Circles on a Flat Torus

An Equal Circle Packing and Its Density

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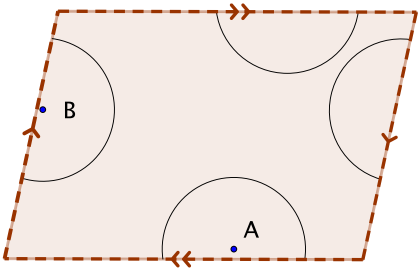
We define its **density** as $\frac{\textit{the area of the circles}}{\textit{the area of the container}}$.

The Flat Torus

Intuitively, a **flat torus** is a parallelogram with opposite sides identified:



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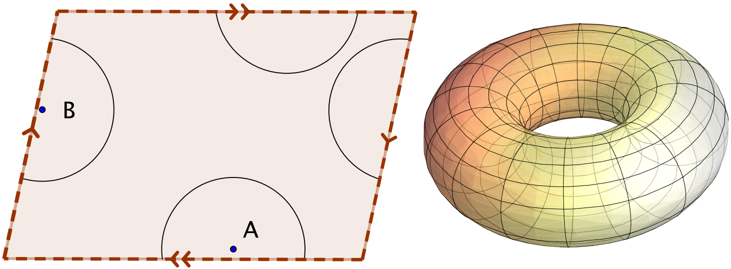
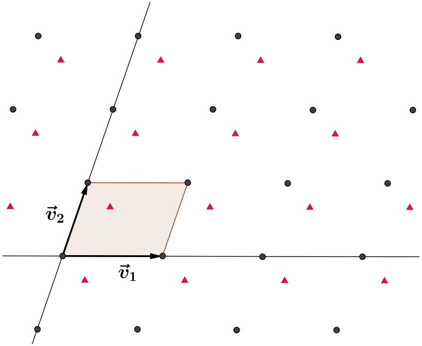


Note that this is not the 3 dimensional torus embedded in $\mathbb{R}^3$.

Equivalence Classes

- We can define a torus by thinking of adjacent sides as vectors. We call these vectors **basis vectors**.



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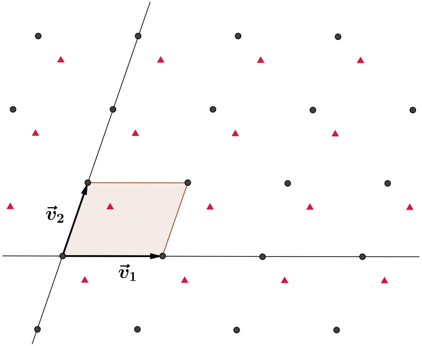

Equivalence Classes

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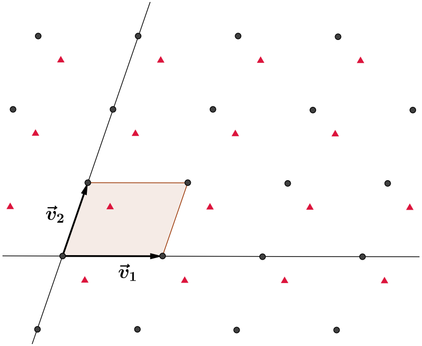
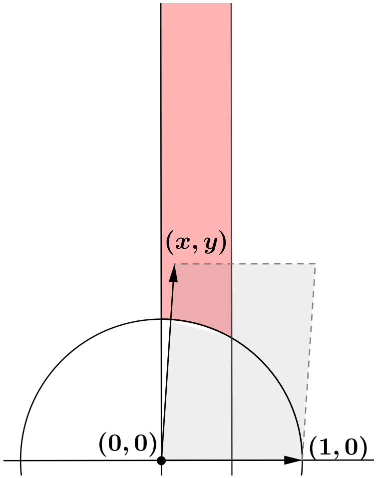
- We call two points that are equivalent different **lifts** of each other.



The Moduli Space

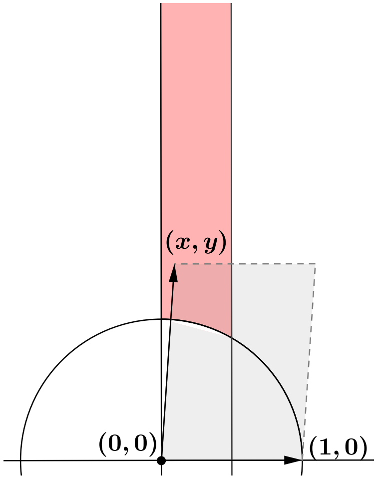
By performing density preserving transformations, we can say any torus is equivalent to one with basis vectors as in the figure.



The Moduli Space

By performing density preserving transformations, we can say any torus is equivalent to one with basis vectors as in the figure.

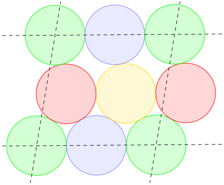
We can therefore associate every torus to a point in the shaded pink region. We call this region the **moduli space** of flat tori.



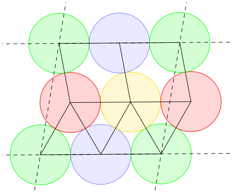
Packing Graphs

Definition

Given a packing, we can associate to it a **equilateral packing graph**, where



Circle Packing



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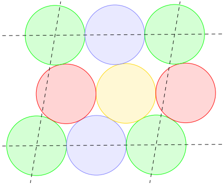

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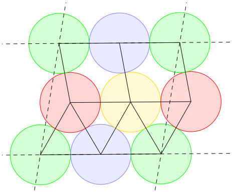

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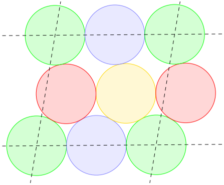
- edges connect centers of tangent circles



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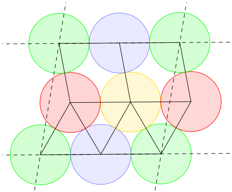

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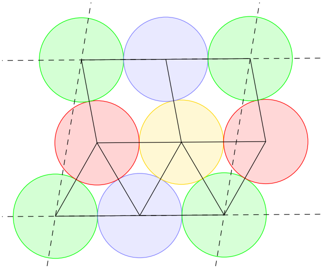

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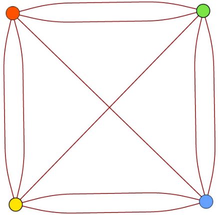

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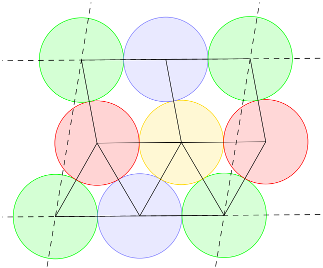

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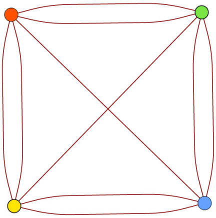

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Given a equilateral packing graph, we can associate to it a **combinatorial (multi)graph**, where:

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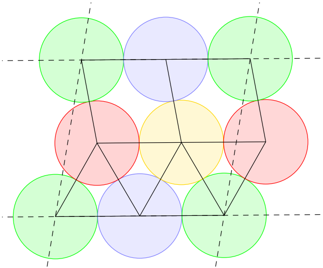
- the number of circles and their shared tangencies are still represented



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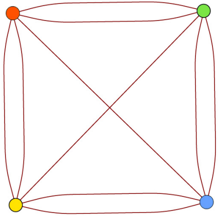

Combinatorial Graph

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Since any optimal packing will have a packing graph associated to it, we can find all optimal packings on flat tori by starting with all possible combinatorial graphs.
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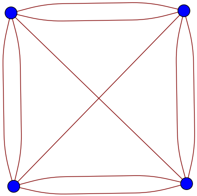
The Combinatorial Graphs we consider

- Using Rigidity Theory it is possible to show that we only need to consider combinatorial graphs with between 7 and 12 edges.



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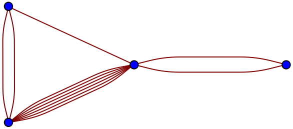

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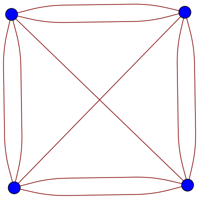

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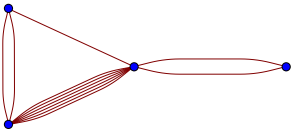

Combinatorial Graph 2

This means we have 825 combinatorial graphs to consider.[1]

[1] we obtained the graphs from Dr. Gordon Royle at The University of Western Australia.

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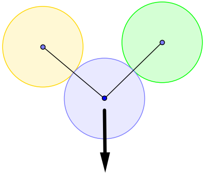

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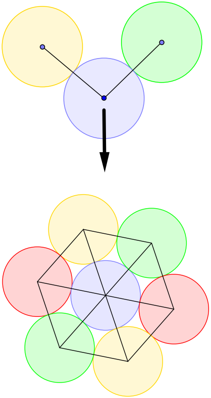
- The maximum degree for each vertex is 6 and only graphs with 12 edges can have a vertex of degree 6.



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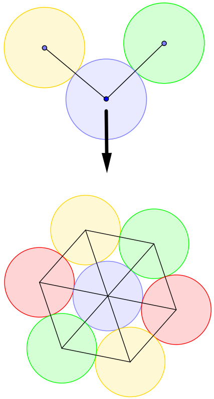

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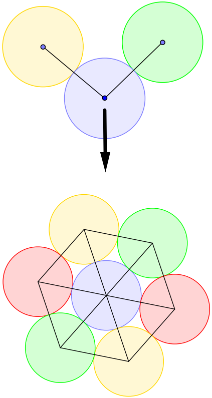
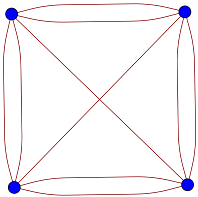


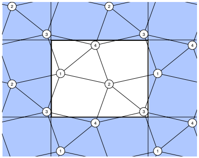
Of the 825, we now have only 14 combinatorial graphs to consider.

Embedding Combinatorial Graphs

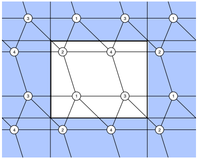
Professor William Dickinson, using techniques from Topological Graph Theory, created a computer program that embeds combinatorial graphs on a flat torus. These graphs are potential packing graphs, but not yet equilateral.



Combinatorial Graph



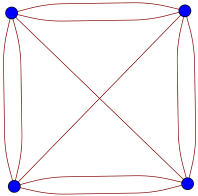
Toroidal Embedding 1



Toroidal Embedding 2

Constructing the Packings Geometrically

Because many of the 14 combinatorial graphs had multiple embeddings, there were 31 potential packing graphs to examine.



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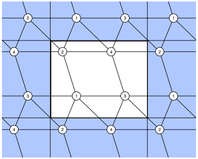

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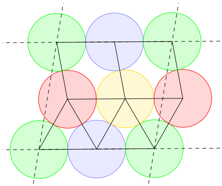
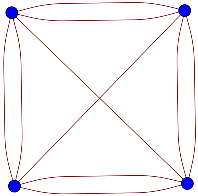


Equilateral Construction

Constructing the Packings Geometrically

Because many of the 14 combinatorial graphs had multiple embeddings, there were 31 potential packing graphs to examine.

We attempted to construct the potential packing graphs as equilateral packing graphs. Using techniques from Rigidity Theory and comparing densities allowed us to determine the 9 distinct equilateral packing graphs that correspond to optimally dense packings.



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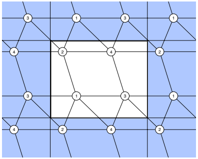

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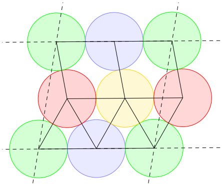
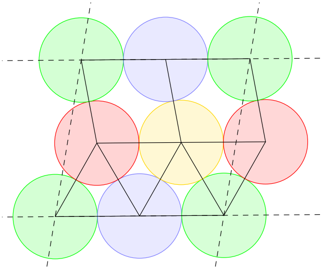


Equilateral Construction

The Correspondence Between a Packing and the Moduli Space

An equilateral packing graph will have a lot of structure, leaving only a small number of free parameters.



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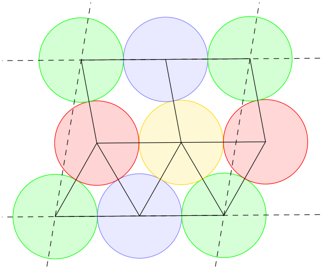
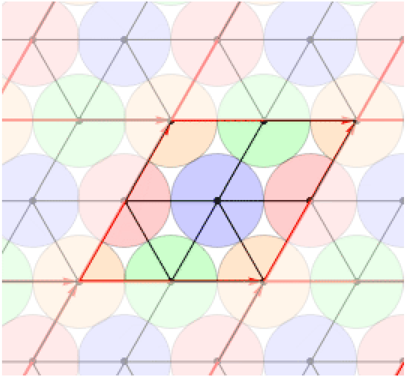
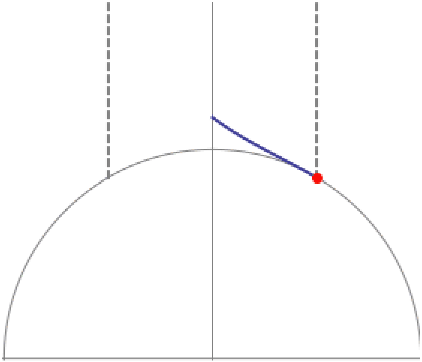


One of our 9 optimal packings

Lifts of the same circle will determine basis vectors, so the equilateral packing graph determines which tori it embeds on.

The Correspondence Between a Packing and the Moduli Space



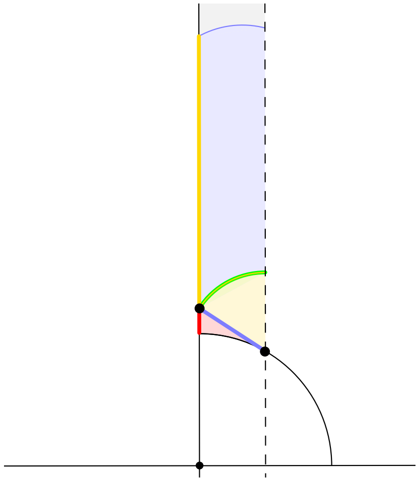


Main Result

Theorem

The globally maximally dense packings of four equal circles on flat tori are given by the packing graphs which occupy the following regions in the moduli space.

Each region corresponds to a packing graph which represents the most dense packing on those tori.



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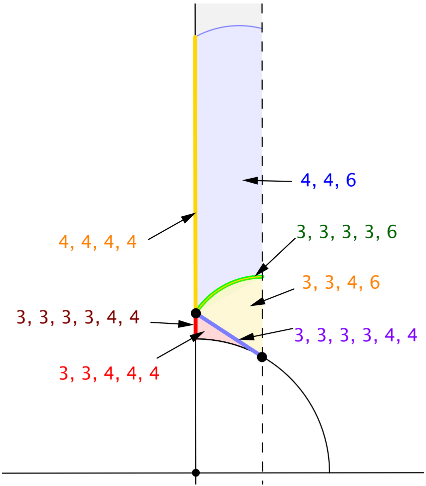
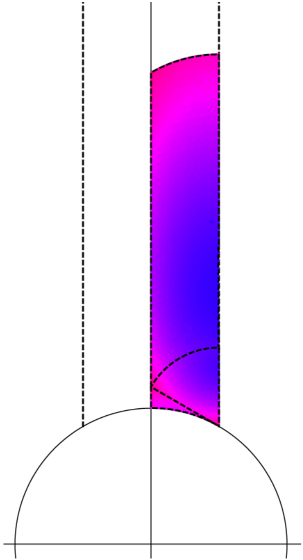


Density in the Moduli Space

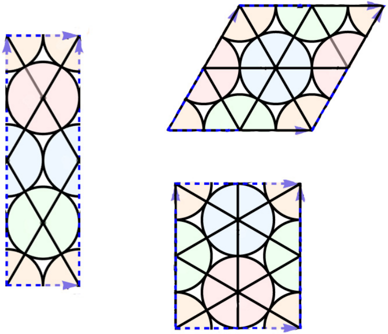
This is a plot of the density of the optimally dense packing for a given point in the moduli space.

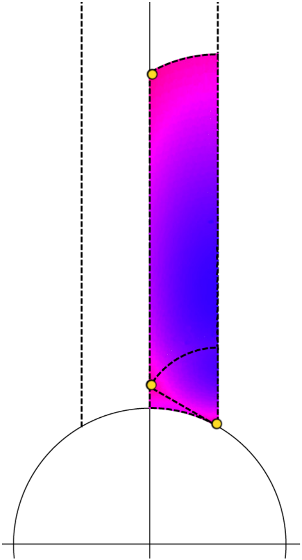
Tori in the pink regions have more dense optimal packings, while tori in the blue regions have less dense optimal packings.



DENSITY IN THE MODULI SPACE

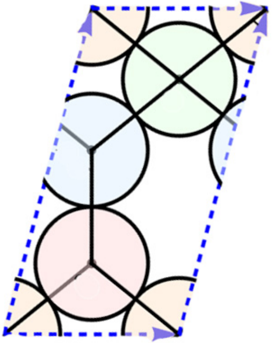
The yellow dots show tori with the highest optimal packing density, $\frac{\pi}{\sqrt{12}} \approx 0.9$.

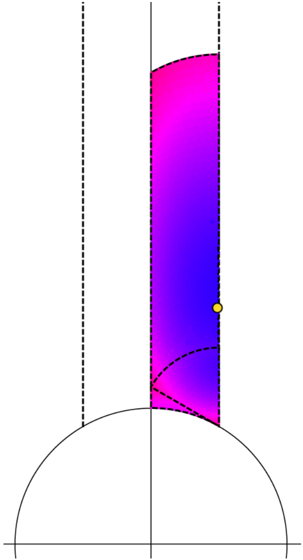




Density in the Moduli Space

The yellow dot shows the torus with the lowest optimal packing density, near 0.7.





Acknowledgements

We would like to thank our advisor, Professor William Dickinson.

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Thank You!