LIMIT OF VORONOI AND DELAUNAY CE: BYGMAC

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This project began with ice cream (see youtube).

1. Voronoi and Delaunay Cells of Points

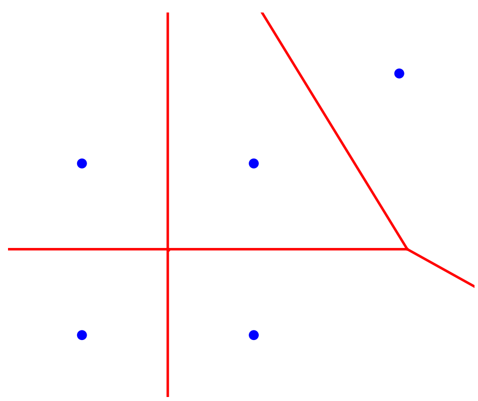
Let $X \subset \mathbb{R}^2$ be a finite collection of points, and let $d(x, y)$ denote the euclidean distance between two points $x, y \in \mathbb{R}^2$.

Definition 1.1. Given a point $x \in X$, define the *Voronoi cell of* x to be

$$\mathrm{Vor}_X(x) = \{y \in \mathbb{R}^2 \mid d(y, x) \leq d(y, x') \text{ for all } x' \in X\}.$$

This is a convex polyhedron of dimension 2. *Convex* means that the line segment between any two points in the set is entirely contained in the set. A *Polyhedron* is the intersection of finitely many half-spaces.

Example 1.2 (Audience Participation: square + point in the plane)**.** How do you draw the Voronoi cells of these points (draw perpendicular bisectors)? Where do they intersect (circumcenter: the circle passing through the 3 points goes through this point)?



Let $B(p, r)$ denote the open disc with center $p \in \mathbb{R}^n$ and radius $r > 0$. We say this disc is *inscribed* with respect to X if $X \cap B(p, r) = \emptyset$ and we say it is *maximal* if no disc containing $B(p, r)$ shares this property. Each inscribed sphere gives a Delaunay cell, defined as follows.

Definition 1.3. Given an inscribed sphere B of a set $X \subset \mathbb{R}^n$, the *Delaunay cell* $Del_X(B)$ is $conv(\overline{B} \cap S)$, where $conv$ denotes the smallest convex set containing the points.

Example 1.4 (Audience Participation: square + point in the plane)**.** How do you draw the Delaunay cells of these points? What do you notice?

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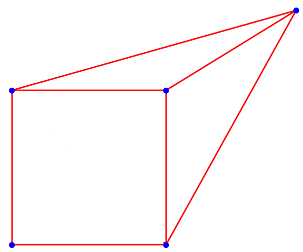


2. Voronoi and Delaunay Cells of Varieties

Throughout let $X \subset \mathbb{R}^2$ be a real plane curve:

$$X = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\},$$

where $f(x, y)$ is a polynomial equation.

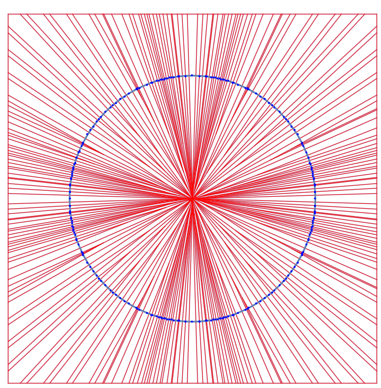
Definition 2.1. Given a point $x \in X$, define the *Voronoi cell of* x to be

$$\mathrm{Vor}_X(x) = \{y \in \mathbb{R}^2 \mid d(y, x) \leq d(y, x') \text{ for all } x' \in X\}.$$

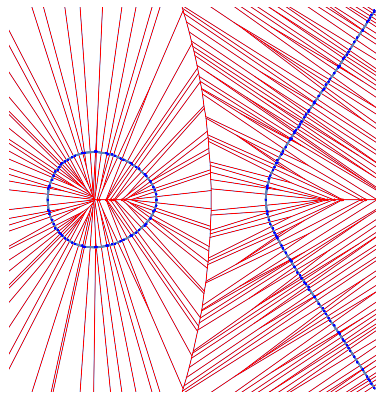
This is a convex semialgebraic set of dimension 1 so long as x is a smooth point of X. It is contained in the *normal space to* X *at* x:

$$N_X(x) = \{u \in \mathbb{R}^n \mid u - x \text{ is perpendicular to the tangent space of } X \text{ at } x\}.$$

Example 2.2. (Audience participation: Voronoi cells of a circle)



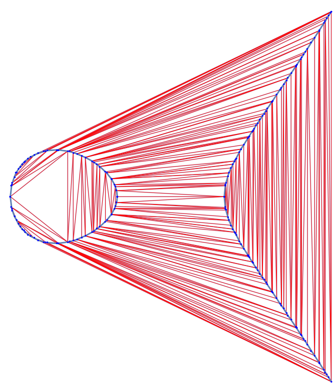
Example 2.3. (Audience participation: Voronoi cells of an elliptic curve) Consider the curve defined by $x(x - 0.5)(x + 0.5) - y^2 = 0$.



Each inscribed sphere in X gives a Delaunay cell, defined as follows.

Definition 2.4. Given an inscribed sphere B of a variety $X \subset \mathbb{R}^n$, the *Delaunay cell $Del_X(B)$* is $conv(\overline{B} \cap S)$.

Example 2.5. (Audience participation: Delaunay cells of an elliptic curve)



3. Limits of Voronoi and Delaunay Cells of Curves

We now wish to study convergence of Voronoi and Delaunay cells. More precisely, given a real algebraic curve X and a sequence of samplings $A_N \subset X$ with $|A_N| = N$, we wish to show that Voronoi (or Delaunay) cells from the Voronoi (or Delaunay) diagrams of the A_N limit to Voronoi (or Delaunay) cells of X. We begin by introducing two notions of convergence which we will use to describe the limits.

The *Hausdorff distance* of two compact sets B_1 and B_2 in $\mathbb{R}^n$ is defined as

$$d_h(B_1, B_2) = \sup \left\{ \sup_{x \in B_1} \inf_{y \in B_2} d(x, y), \sup_{y \in B_2} \inf_{x \in B_1} d(x, y) \right\}.$$

A sequence $\{B_\nu\}_{\nu \in \mathbb{N}}$ of compact sets is *Hausdorff convergent* to B if $d_h(B, B_\nu) \to 0$ as $\nu \to \infty$. Given a point $x \in \mathbb{R}^{\ltimes}$ and a closed set $B \subset \mathbb{R}^n$, let

$$d_w(x, B_1) = \inf_{b \in B} d(x, b).$$

Given a sequence of closed sets B_i, we say they are *Wijsman convergent* to B if for every $x \in \mathbb{R}^n$, we have that

$$d_w(x, A_i) \to d_w(x, A).$$

We use Wijsman convergence as a variation of Hausdorff convergence which is well suited for unbounded sets.

An ϵ-*approximation* of a real algebraic variety X is a discrete subset $A_\epsilon \subset X$ such that for all $y \in X$ there exists an $x \in A_\epsilon$ so that $d(y,x) \leq \epsilon$. By definition, a sequence of ϵ-approximations of X is Hausdorff or Wijsman convergent to X (depending upon which is appropriate). We need these two notions of convergence because Delaunay cells are always compact, while Voronoi cells may be unbounded.

We now study convergence of Delaunay cells of X, and introduce a condition on real algebraic varieties which ensures that the Delaunay cells are simplices.

Definition 3.1. We say that a real algebraic variety $X \subset \mathbb{R}^n$ is *Delaunay-generic* if X does not meet any d-dimensional inscribed sphere greater than $d+2$ points.

Example 3.2. (Audience participation) Can anyone provide an example of a variety that is not Deluanay-generic?

Theorem 3.3 (B-Weinstein)**.**

(1) Let $X \subset \mathbb{R}^n$ be a Delaunay-generic real algebraic curve, and let A_ϵ be a sequence of ϵ-approximations of X. Every maximal Delaunay cell is the Hausdorff limit of a sequence of Delaunay cells of A_ϵ.

(2) Let X be a compact and smooth curve in $\mathbb{R}^2$ and $\{A_\epsilon\}_{\epsilon \searrow 0}$ be a sequence of ϵ-approximations. Then every Voronoi cell is the Wijsman limit of a sequence of Voronoi cells of A_ϵ.

4. Future Directions

(1) Eliminate compact and smooth from the assumptions in the theorem.

(2) Investigate which metric features of a real plane curve (e.g., curvature, singularities, normal direction, bottlenecks, medial axis, reach, evolute) can be detected from a Voronoi decomposition.

(3) Study degrees of critical curvature, bottlenecks.

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