From Polynomials to Metric Graphs

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Polynomials and Polytopes
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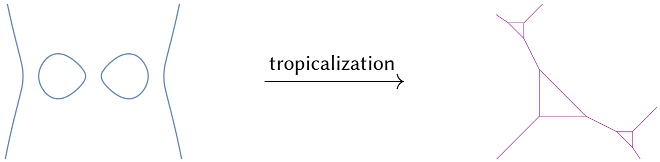
Tropical Geometry

Goal: Turn questions about algebraic varieties into questions about polyhedral complexes.



Why: The polyhedral complex retains some information about the original variety that we are interested in.

Valuation

Definition

A valuation is a function $v : K \to \mathbb{R} \cup \{\infty\}$ satisfying:

1. $v(a) = \infty$ if and only if $a = 0$,
2. $v(ab) = v(a) + v(b)$
3. $v(a + b) \geq \min\{v(a), v(b)\}$

Let K be an algebraically closed field which is complete with respect to a non-trivial valuation $v : K^* \to \mathbb{R}$.

Example

The Puiseux series: $\mathbb{C}\{\{t\}\} = \{f = \sum_{k=k_0}^{\infty} c_k t^{k/n} \mid c_k \in \mathbb{C}, n \in \mathbb{Z}\}$
Valuation: $v(f) = k/n$, where k is the smallest index with $c_k \neq 0$.

Tropicalization of Algebraic Varieties

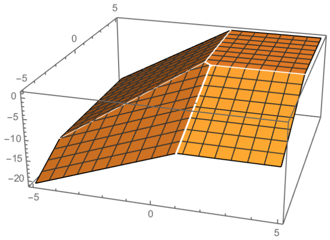
Let $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be an ideal and $V(I) = X \subset (K^*)^n$ be an algebraic subvariety of the torus. Then the tropicalization of X is:

$$\mathrm{Trop}(X) = \{(v(x_1), \ldots, v(x_n)) \in \mathbb{R}^n \mid (x_1, \ldots, x_n) \in X = V(I)\ \}$$

If the variety X is a hypersurface defined by a single polynomial $f = \sum c_a x^a$, then this is the same as the collection of points $x \in \mathbb{R}^n$ where

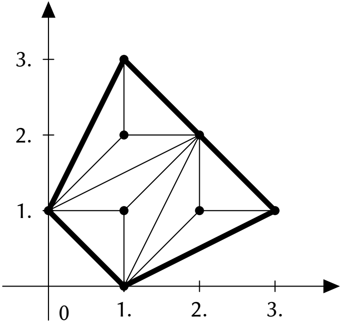
$$min_{a \in \mathbb{Z}^n}(v(c_a) + x \cdot a)$$

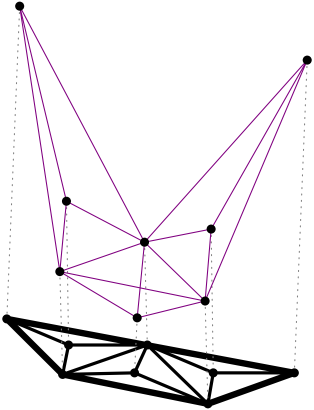
is attained twice.



Tropicalizing Plane Curves: Newton Polytope and Subdivision

$$f(x,y,z) = -2^8x^3y - 2x^2y^2 - 2^8xy^3 - 2^3x^2y - 2^3xy^2 - xy - 2x^3 - 2y$$





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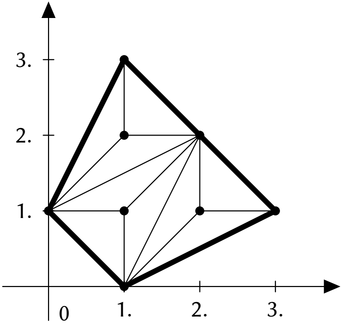
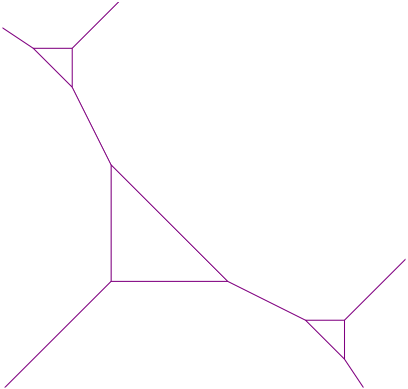




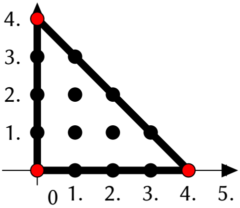
Example: Depends on embedding

$$f(X,Y,Z) = -2^8X^3Y - 2X^2Y^2 - 2^8XY^3 - 2^3X^2YZ - 2^3XY^2Z - XYZ^2 - 2X^3Z - 2YZ^3$$

Change coordinates:

$X = x + y - z,\ Y = x - y - z,\ Z = -5x - 5y - 7z$

$$f = 41x^4 + 1530x^3y + 3508x^3z + 1424x^2y^2 + 2490x^2yz - 2274x^2z^2 + 470xy^3 + 680xy^2z - 930xyz^2 + 772xz^3 + 535y^4 - 350y^3z - 1960y^2z^2 - 3090yz^3 - 2047z^4,$$



and now the tropicalization reveals nothing.

Question: How do we find the best tropicalization?

What is a metric graph?

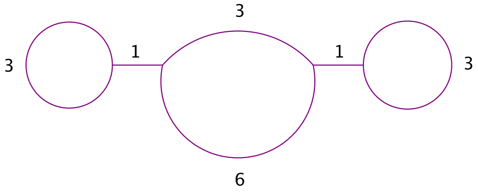
Definition

A metric graph is a triple (G, l, w) where:

- G is a connected graph,
- $l : E(G) \to \mathbb{R}_{>0}$ gives the edge lengths,
- $w : V(G) \to \mathbb{Z}_{\geq 0}$ is a weight function on the vertices.

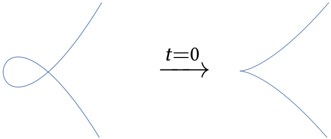
We require that every weight 0 vertex has degree at least 3.

The genus is $g(G) + \sum_{v \in V} w(v)$.



Semistable Models

If X is a curve defined over the Puiseux series, we will now think of X as a family of curves X_t with parameter t which is nice for $t \neq 0$ and X_0 is possibly singular.



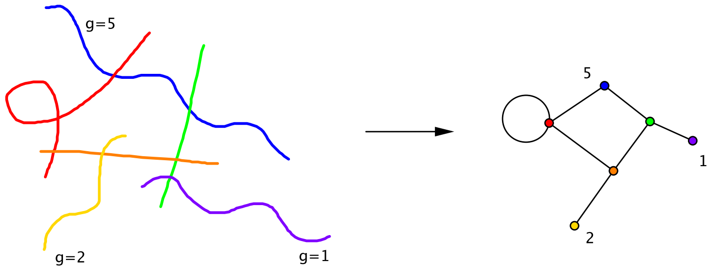
We need the special fiber X_0 to be semistable: has at worst nodal singualrities.

Theorem (Semistable reduction theorem)

We can always "replace" the (potentially non-semistable) curve X_0 obtained by setting $t = 0$ with another semistable curve.

Dual Graph

The dual graph to a semistable curve with irreducible components $C_1, \ldots, C_n$ is the graph with vertices v_i corresponding to the components, and an edge whenever the components intersect. If C_i has genus g_i, then $w(v_i) := g_i$.

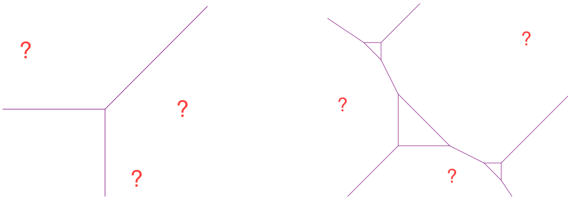


Tropicalization and Metric Graphs

How does this relate back to tropicalization?

If one has a faithful tropicalization, the structure seen in the tropicalization is the same as in this metric graph [BPR13].

However, it can be difficult to certify faithfulness. It is not known how to certify a faithful tropicalization in general [BPR16, BBCar].

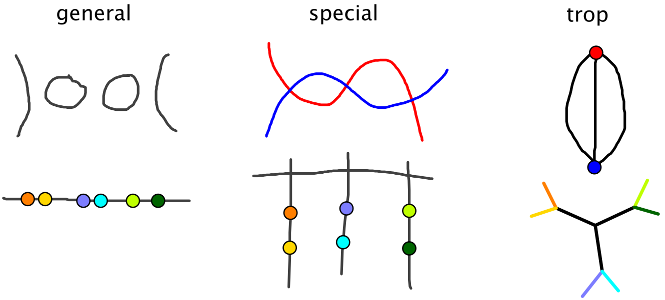


My goal is to be able to find this metric graph in an algorithmic way just from polynomial equations defining a curve.

Hyperelliptic Curves and Ramification Points

hyperelliptic curve $\leftrightarrow$ $y^2 = f(x)$, $\deg(f) = 2g + 2$ $\leftrightarrow$ double cover of $\mathbb{P}^1$ ramified at roots of f

$$y^2 = (x - 1)(x - 1 - t)(x - 2)(x - 2 - t)(x - 3)(x - 3 - t)$$



Algorithm for $\mathcal{M}_{0,2g+2} \to \mathcal{M}_{0,2g+2}^{\text{trop}}$

This algorithm is described in [RSS14].

$$p = (1, 1+t, 2, 2+t, 3, 3+t)$$

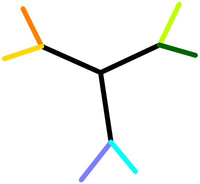
1. Let w_{ij} be a vector of $\binom{n}{2}$ pairwise differences of the points p_k.

$$w = (t, 1, 1+t, 2, 2+t, 1-t, 1, 2-t, 2, t, 1, 1+t, 1-t, 1, t)$$

2. The vector $d_{ij} := N - 2v(w_{ij})$ for large N is a distance vector on a tree: d_{ij} is the distance between leaves i and j.

$$d = (2, 4, 4, 4, 4, 4, 4, 4, 4, 2, 4, 4, 4, 4, 2)$$

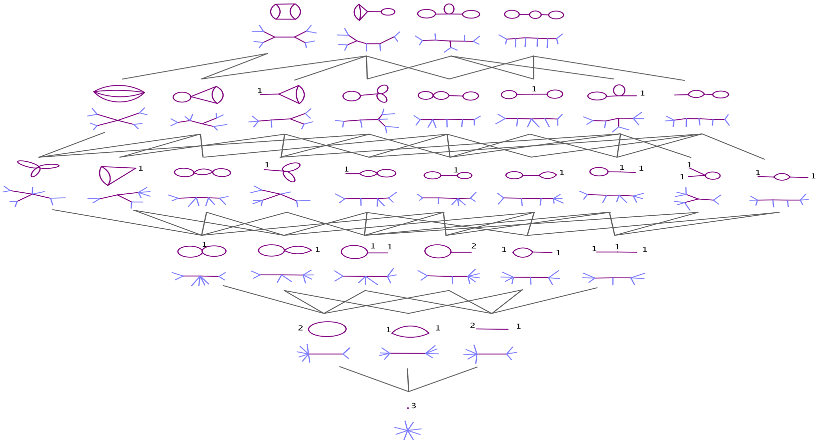
3. Use the neighbor joining algorithm to reconstruct the tree.



Tropicalizations of Hyperelliptic Curves

Theorem

For every tree, there is exactly one metric graph which is a degree 2 admissible cover of that tree. This gives an algorithm for abstract tropicalization of hyperelliptic curves.



Conclusion

Next:

1. Tropicalizing plane quartics.
2. Tropicalizing curves $y^p = f(x)$.

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