Tropical Geometry Is Fun
Introductory Lecture

Madeline Brandt

June 11 2018

Introduction

Tropical geometry is a new subject which creates a bridge between the two islands of algebraic geometry and combinatorics. It has many fascinating connections to other areas as well. The aim of this lecture is to set up all background needed to study tropical geometry. We will cover the material in Chapter 2, Sections 1-4 of the Maclagan-Sturmfels book [MS15].

2.1: Fields

Tropicalization is a process we apply to varieties over a field which are equipped with a function called a valuation, so we begin here to start the study of tropical geometry.

Definition 1. A **valuation** on a field K is a function $\nu : K \to \mathbb{R} \cup \{\infty\}$ satisfying

1. $\nu(a) = \infty$ if and only if $a = 0$,
2. $\nu(ab) = \nu(a) + \nu(b)$,
3. $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$.

The image of ν is denoted by Γ_{val}, and is called the **value group**. This is an additive group and we assume it contains 1.

Lemma 2. *If* $\nu(a) \neq \nu(b)$, *then* $\nu(a + b) = min(a + b)$.

Question 3. Prove Lemma 2.

We denote by R the set of all elements with nonnegative valuation. It is a local ring with maximal ideal $\mathfrak{m}$ given by all elements with positive valuation. The quotient ring is denoted by $\Bbbk$ and it is called the **residue field**.

Example 4. Every field has a trivial valuation sending $\nu(K^*) = 0$.

Example 5. (p-adic valuation) The p-adic valuation on $\mathbb{Q}$ is given by a prime p, and the valuation of a rational number

$$\frac{ap^l}{bp^k}$$

With $p \nmid a, b$ and $(a, b) = 1$ is $l - k$. The local ring R is the localization of $\mathbb{Z}$ at $< p >$, and the residue field is $\mathbb{F}_p$.

Example 6. (Puiseaux series) The Puiseaux series are the formal power series with rational exponents and coefficients in $\mathbb{C}$:

$$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$$

for a_i an increasing sequence of rational numbers which have a common denominator. The valuation is given by taking $\nu(c) = a_1$. The Puiseaux series are algebraically closed.

A **splitting** of a valuation is a homomorphism $\phi : \Gamma_{\text{val}} \to K^*$ such that $\nu(\phi(w)) = w$. The element $\phi(w)$ is denoted t^w, and t is called a **uniformizer** for K.

Question 7. Show that the residue field of $\mathbb{C}\{\{t\}\}$ is $\mathbb{C}$.

Question 8. What is the residue field of $\mathbb{Q}$ with the p-adic valuation?

2.3: Polyhedral Geometry

Polyhedral geometry is a deep and beautiful subject in discrete mathematics. Here we just give the basics of what we will need for this seminar. See [Zie95] for more about polyhedral geometry.

Definition 9. A set $X \subset \mathbb{R}^n$ is **convex** if for any two points in the set, the line segment between them is also contained in the set. The **convex hull** $\mathrm{conv}(U)$ of a subset $U \subset \mathbb{R}^n$ is the smallest convex set containing U. A **polytope** is a convex set which is expressible as the convex hull of finitely many points.

A **polyhedral cone** C in $\mathbb{R}^n$ is the positive hull of a finite subset of $\mathbb{R}^n$:

$$C = \mathrm{pos}(v_1, \dots, v_r) := \left\{ \sum_{i=1}^{r} \lambda_i v_i \,\middle|\, \lambda_i \geq 0 \right\}.$$

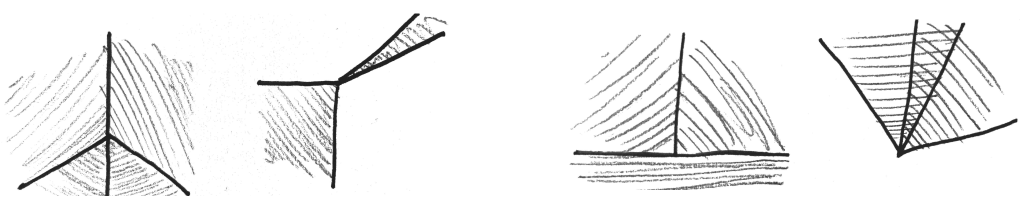
A **polyhedron** is the intersection of finitely many half spaces. Bounded polyhedra are polytopes: these are equivalent ways to define them.

A **face** of a cone C is determined by a linear functional $w \in \mathbb{R}^n$, by selecting the points along which the linear functional is minimized:

$$\mathrm{face}_w(C) = \{x \in C \mid wx \leq wy \text{ for all } y \in C\}.$$

A face which is not contained in any larger proper face is called a **facet**.

Definition 10. A **polyhedral fan** is a collection $\mathcal{F}$ of polyhedral cones such that every face of a cone is in the fan, and the intersection of any two cones in the fan is a face of each. For some examples and nonexamples, see Figure 1.



(a) These keep you cool in the tropics.

(b) “I’m not a fan.”

Figure 1: Examples and nonexamples of polyhedral fans.

Definition 11. A **polyhedral complex** Is a collection Σ of polyhedra such that if $P \in \Sigma$ then every face of P is also in Σ, and if P and Q are polyhedra in Σ then their intersection is either empty or also a face of both P and Q. See Figure 2 for an example.

The **support** $|\Sigma|$ of Σ is the union of all of the faces of Σ.

There is one last thing we will need from polyhedral geometry, and that is the notion of a regular subdivision.

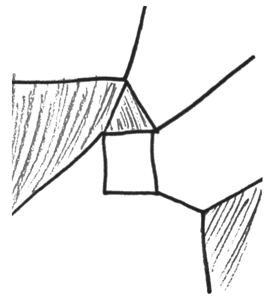


Figure 2: An example of a polyhedral complex.

Definition 12. Let $v_1, \dots, v_r$ be an ordered list of vectors in $\mathbb{R}^n$. We fix a **weight vector** $w = (w_1, \dots, w_r)$ in $\mathbb{R}^r$ assigning a weight to each vector. Consider the polytope in $\mathbb{R}^{n+1}$ defined by $P = \text{conv}((v_1, w_1), \dots, (v_n, w_n))$. The **regular subdivision** of $v_1, \dots, v_r$ is the polyhedral complex on the points $v_1, \dots, v_r$ whose faces are the faces of P which are "visible from beneath the polytope". More preceisely, the faces σ are the sets for which there exists $c \in \mathbb{R}^n$ with $c \cdot v_i = w_i$ for $i \in \sigma$ and $c \cdot v_i < w_i$ for $i \notin \sigma$.

Question 13. Find all regular subdivisions of $\{(0,0), (0,1), (1,0), (1,1)\}$ and give examples of weight vectors which give these subdivisions.

Question 14. A fan or a polyhedral complex is **pure of dimension** d if every maximal face has the same dimension, d. Give examples of fans which are and are not pure.

Question 15. A pure, d-dimensional polyhedral complex in $\mathbb{R}^n$ is **connected through codimension 1** if for any two d dimensional cells, there is a chain of d-dimensional cells $P = P_1, \dots, P_s = P'$ for which P_i and P_{i+1} share a common facet F_i. Give an example of a polyhedral complex which is not connected through codimension 1 in $\mathbb{R}^2$.

Question 16. Are k-skeleta of polyhedra connected through codimension 1?

2.4: Gröbner Bases

Gröbner Bases are an exceedingly useful and miraculous computational tool. Here, we will restrict our discussion just to what is necessary for tropical geometry. If this is your first time seeing Gröbner Bases, my condolences.

Let K be an arbitrary field, with valuation (possibly the trivial valuation). Consider the ring $K[x_1, \dots, x_n]$ of polynomials over K.

Definition 17. Given a polynomial

$$f = \sum_{u \in \mathbb{N}^n} c_u x^u,$$

we define its **tropicalization** to be the real valued function on $\mathbb{R}^n$ that is obtained by replacing each c_u by its valuation and preforming all additions and multiplications in the semiring $(\mathbb{R}, \oplus, \otimes)$:

$$\operatorname{trop}(f)(w) = \min_{u \in \mathbb{N}^n} (\operatorname{val}(c_u) + u \cdot w)$$

Definition 18. Let $w \mapsto t^w$ be a splitting of the valuation on K. The **initial form** of f with respect to w is

$$\operatorname{in}_w(f) = \sum_{u : \operatorname{val}(c_u) + w \cdot u = \operatorname{trop}(f)(w)} \overline{t^{-\operatorname{val}(c_u)} c_u} x^u.$$

For a homogeneous ideal $I \subset K[x_1, \dots, x_n]$, the **initial ideal** is the set

$$\operatorname{in}_w(I) = \langle \operatorname{in}_w(f) \mid f \in I \rangle.$$

Definition 19. Let $G \subset I$ be finite. We call G a **Gröbner Basis for I** with respect to $w \in \mathbb{R}^n$ if

$$\operatorname{in}_w(I) = \langle \operatorname{in}_w(g) \mid g \in G \rangle.$$

Lemma 20. *Gröbner bases exist.*

Question 21. Let $K = \mathbb{Q}$ with the 2 adic valuation. Let

$$I = +2y - 3x, 3y - 4z + 5w\rangle.$$

What is $V(I)$ and where does it live? For $w = (0, 0, 0, 0)$ and $w = (1, 0, 0, 1)$ compute $\operatorname{in}_w(I)$. What are all of the initial ideals of I, and which collection of w produce each one?

References

[MS15] D. Maclagan and B. Sturmfels, **Introduction to tropical geometry:**, Graduate Studies in Mathematics, American Mathematical Society, 2015.

[Zie95] Günter M. Ziegler, **Lectures on polytopes**, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995.