# Tropical Geometry of Curves

Madeline Brandt

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$$\begin{array}{ccc} \text{Algebraic Geometry} & & \text{Combinatorics} \\ \textit{Algebraic Varieties} & \xrightarrow{\text{tropicalization}} & \textit{Tropical varieties /} \\ \{x \in K^n \mid f_1(x) = \cdots = f_l(x) = 0\} & & \textit{polyhedral complexes} \end{array}$$

**Today**: Tropical geometry of curves.

- I will define two different objects which one could call a "tropical curve".
- Say how these are related.
- My results on how to compute these objects.

## 1 Geometry over Non-Archimedean fields

Tropical geometry deals with varieties over **non-Archimedean fields**. Let's see how they differ from the fields we are used to dealing with.

**Definition 1.1.** $(K, |\cdot|)$ is an **Archimedean field** if it satisfies the **Archimedean axiom**: for any $x \in K^*$, there is an $n \in \mathbb{N}$ such that $|nx| > 1$.

While this axiom may feel natural and familiar, $\mathbb{R}$ and $\mathbb{C}$ are essentially the only fields satisfying this axiom.

A **non-Archimedean field** is one with a norm $|\cdot|$ which fails this axiom. It comes with a function called the **valuation**: $\mathrm{val}_K : K \to \mathbb{R} \cup \{\infty\}$ defined by:

$$\mathrm{val}_K(a) = -\log(|a|), \quad \mathrm{val}_K(0) = \infty.$$

**Example 1.2.** The **Puiseux series** $\mathbb{C}\{\{t\}\}$ is:

$$\left\{ c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \cdots \;\middle|\; \begin{array}{c} c_1 \neq 0,\ c_i \in \mathbb{C}, \\ a_i \text{ an } \nearrow \text{ sequence in } \mathbb{Q} \\ \text{w/ common denominator} \end{array} \right\} \cup \{0\}.$$

The valuation sends $c(t) \mapsto a_1$. The norm is $|c(t)| = \epsilon^{\mathrm{val}_K(c(t))}$ for $\epsilon \in (0, 1)$

# 2 Embedded tropicalization

I will now describe how to find the **embedded tropicalization** of a variety (hypersurface) over a non-Archimedean field $K$.

**Definition 2.1.** Given a polynomial

$$f(x) = \sum_{a \in \mathbb{N}^n} c_a x^a \quad \in K[x_1, \ldots, x_n],$$

Its **tropicalization** $\mathtt{trop}(f) : \mathbb{R}^n \to \mathbb{R}$ is (obtained by replacing each $c_a$ by its valuation and preforming all additions and multiplications in the **tropical semiring** $(\mathbb{R}, \oplus, \otimes)$):

$$\mathrm{trop}(f)(x) = \min_{a \in \mathbb{N}^n} \left( \mathrm{val}(c_a) + a \cdot x \right).$$

Just as we can associate a variety to $f$, we will see how to make a tropical variety associated to $\mathrm{trop}(f)$.

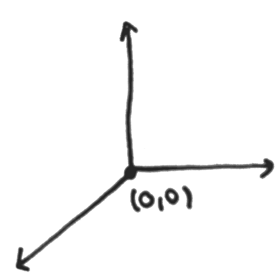
**Definition 2.2.** The **tropical hypersurface** associated to $\mathrm{trop}(f)$ is the set

$$\{x \in \mathbb{R}^n \mid \text{the minimum in } \mathrm{trop}(f)(x) \text{ is achieved at least twice}\}$$

**Example 2.3.** Here we compute the **tropical line**. Let $f = x + y + 1 \in \mathbb{C}\{\{t\}\}[x, y]$. Then,

$$\mathrm{trop}(f)(x, y) = \min(x, y, 0).$$

So, where is this minimum achieved twice? We can break this down in to 3 cases.



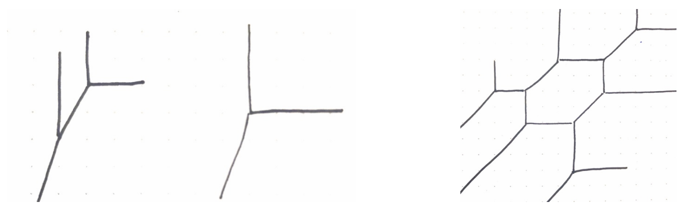
**Problem**: Embedded tropicalizations are not invariant under coordinate change, and can change quite drastically.

**Question 2.4.** How do we associate an intrinsic tropical object to a curve?

**Example 2.5.** Tropicalizing an elliptic curve in normal form $y^2 = x^3 + ax + b$ can only give trees (pictured below).

**Theorem 2.6** (Chan-Sturmfels)**.** *Every elliptic curve with* $\mathrm{val}_K(j) < 0$ *has an embedding such that the embedded tropicalization is a honeycomb.*

The honeycomb "reveals" that the curve has genus 1 and the valuation of j.



# 3 Abstract tropicalization

$$\text{Complicated curve} \xrightarrow{\text{degenerates}} \text{union of simpler curves}$$

We can encode **families of curves** using algebraic geometry over N.A. fields.

**Notation**: Let $K$ be a non-Archimedean field.

| | |
|---|---|
| $R = \{f \in K \mid \mathrm{val}_K(f) \geq 0\}$ | **Valuation ring**: local ring |
| $\mathfrak{m} = \{f \in K \mid \mathrm{val}_K(f) > 0\}$ | unique maximal ideal |
| $k = R/\mathfrak{m}$ | **residue field** |

**Comments on Spec$R$**: Spec$(R)$ is a topological space with two points, one corresponding to the $0$ ideal and the other corresponding to $\mathfrak{m}$. The point $\mathfrak{m}$ is closed, and the closure of the point $0$ is all of Spec$(R)$.

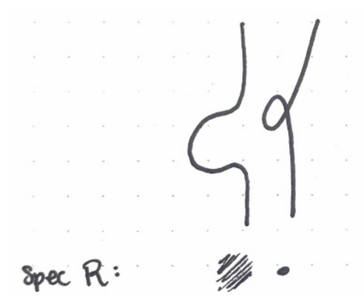
**Definition 3.1.** Let $X$ be a smooth curve over $K$. A **model** $\mathcal{X}$ for $X$ is a flat and finite type scheme over $R$ whose **generic fiber** $\mathcal{X} \times_R K$ (fiber over (0), $K$ = residue field of the point 0) is isomorphic to $X$. We call this model **semistable** if it is proper and the **special fiber** $\mathcal{X} \times_R k$ (fiber over the maximal ideal) satisfies:

1. It is reduced, connected, and only has nodal singularities;
2. every rational (isomorphic to $\mathbb{P}^1$) component meets the rest of the curve in at least 2 singular points.

[the adjectives flat, finite type, proper are technical assumptions that ensure the family is "nice": e.g., each fiber is of the same dimension, points in general fiber have unique closures to points in special fiber
irred component: not the union of two proper alg subsets]

**Remark 3.2.** In the abstract, we are always guaranteed that a semistable model exists. In practice, it can be difficult to find.

**Example 3.3.** Consider the curve $y^2 = x^3 + x^2 + t^4$ over $\mathbb{C}\{\{t\}\}$. This is a smooth elliptic curve for $t \neq 0$. However, when $t = 0$, we have the curve defined by the equation $y^2 = (x+1)x^2$.

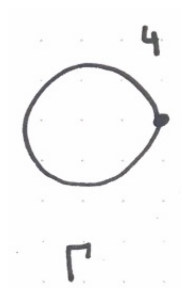



**Definition 3.4.** The **abstract tropicalization** $\Gamma$ of $X$ is a metric graph with:

1. **Vertices**: corresponding to the irreducible components of $\mathcal{X}_k$
2. **Edges**: corresponding to nodes.
3. **Vertex Weights**: we add weights to the vertices by assigning to each vertex the geometric **genus** of the corresponding component.

4. **Edge Lengths**: We add edge lengths in the following way. Given an edge corresponding to a node $q$ between two components $X_i$ and $X_j$, the completion of the local ring $\mathcal{O}_{\mathcal{X},q}$ is isomorphic to $R[[x,y]]/(xy-f)$ where $\mathrm{val}_K(f) > 0$. Then, we define the length of the edge $e_{ij}$ to be $v(f)$.

**Example 3.5.** Consider the curve $y^2 = x^3 + x^2 + t^4$ over $\mathbb{C}\{\{t\}\}$. This is a curve with self-intersection. This is already a semistable model, and the abstract tropicalization is a cycle of length 4. [The formal completion at $(0,0)$ is $R[u,v]/(uv - t^4)$, where $x' = x\sqrt{x+1}$, $u = y - x'$, and $v = y + x'$]




# 4 How are these related?

Embedded tropicalization for a "good" embedding gives the abstract trop.

Finding (and certifying) these good embeddings is difficult.

# 5 Computing abstract tropicalizations

**Question 5.1.** How do we compute the **abstract tropicalization** of a curve?

The problem of computing the abstract tropicalization has been studied in several classes of curves and by multiple approaches.

1. genus 1: the answer has been known for some time; one simply takes the valuation of the j-invariant of the curve and if it is negative, then the abstract tropicalization will be a cycle of length negative of this valuation.

2. genus 2: this was done systematically by studying the ramification data [Ren-Sam-Sturmfels], using Igusa invariants [Helminck], and by finding good embeddings to do the embedded tropicalization in [Cueto-Markwig].

**Theorem 5.2.** *There is an algorithm for computing the abstract tropicalization of*

- *hyperelliptic curves ($y^2 = g(x)$) [Bolognese-B-Chua]*
- *supelliptic curves ($y^m = g(x)$) [B-Helminck]*

The procedure can be summarized as follows:

1. Superelliptic and hyperelliptic curves admit a 2:1 map to $\mathbb{P}^1$ ($(x, y) \mapsto x$), ramified over the roots of $g$. This entire situation tropicalizes to a 2:1 map of metric graphs $\Gamma \to T$.
2. $T$ is the tropicalization of $\mathbb{P}^1$ together with the marked points. This is a tree, which is easy to calculate.
3. For hyperelliptic curves, this tree completely determines the structure of the graph. For superelliptic curves, additional data is provided by studying divisors on the curve.

# 6 Conclusion

We looked at 2 different ways to define a tropical curve / think about geometry over a non-Archimedean field:

1. Embedded tropicalization: easy to compute, not intrinsic
2. Abstract tropicalization: intrinsic but hard to compute

And saw how these objects relate to one another. We also studied the problem of how to compute abstract tropicalizations, and saw that there is an algorithm to compute them for superelliptic curves $y^n = g(x)$.