Math 255 Lectures: 6.1, 6.2

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6.1: Holomorphic differentials

Given a lattice Λ in $\mathbb{C}$, we have associated a nonsingular cubic curve C_Λ in $\mathbb{P}_2$ defined by the equation

$$y^2z = 4x^3 - g_2(\Lambda)xz^2 - g_3(\Lambda)z^3.$$

Now, we wish to know whether given a curve C_Λ, can we recover the lattice Λ? In order to do this, we now introduce some new concepts.

Definition 1. A **piecewise smooth path** in a Riemann surface S is a continuous map γ from a closed interval $[a, b] \subset \mathbb{R}$ to S such that if $\phi : U \to V$ is a holomorphic chart on an onpen subset U of S and $[c, d] \subset \gamma^{-1}(U)$ then

$$\phi \circ \gamma : [c, d] \to V$$

is a piecewise smooth path in $V \subset \mathbb{C}$. The path is **closed** if $\gamma(a) = \gamma(b)$.

Definition 2. A **meromorphic function** on a Riemann surface S is a function $f : S \to \mathbb{P}_1$ which is holomorphic (in the sense of Riemann surfaces) and is not identically ∞ on any connected compoenent of S.

Remark 3. A holomorphic function $f : S \to \mathbb{C}$ on a compact Riemann surface S is a constant. However, there are many interesting meromorphic functions $f : S \to \mathbb{P}_1$.

Example 4. Let C be an irreducible projective curve defined by $P[x, y, z] = 0$. By a **rational function** on $C - \text{Sing}(C)$, we mean a meromorphic function on $C - \text{Sing}(C)$ of the form

$$[x : y : z] \mapsto \frac{S(x, y, z)}{T(x, y, z)}$$

where S and T are homogeneous polynomials **of the same degree** and T does not vanish everywhere on C. If S and T both have degree k, then

$$\frac{S(\lambda x, \lambda y, \lambda z)}{T(\lambda x, \lambda y, \lambda z)} = \frac{\lambda^k S(x, y, z)}{\lambda^k T(x, y, z)}$$

so the function is well-defined.

Now, we will introduce meromorphic differentials, which we plan to integrate later.

Definition 5. Let f and g be meromorphic functions on a Riemann surface S. Then we say that

$$f\ dg$$

is a **meromorphic differential** on S, and if f' and g' are also meromorphic functions on S, we say that

$$f\ dg = f'\ dg'$$

if for every holomorphic chart $\phi : U \to V$ we have

$$(f \circ \phi^{-1})(g \circ \phi^{-1})' = (f' \circ \phi^{-1})(g' \circ \phi^{-1})'.$$

Remark 6. There are some technical comments in the book following this definition, including an alternative definition of "meromorphic differential". We skip this now in the interest of time.

Definition 7. We say that the meromorphic differential $f\ dg$ has a pole at a point p in S if whenever $\phi : U \to V$ is a holomorphic chart on $U \ni p$, we have that $\phi(p)$ is a pole of the meromorphic function $(f \circ \phi^{-1})(g \circ \phi^{-1})'$. We call $f\ dg$ a **holomorphic differential** if it has no poles.

As promised, we now integrate.

Definition 8. If $f\ dg$ is a holomorphic differential on S then the integral of $f\ dg$ along a piecewise-smooth path $\gamma : [a, b] \to S$ is

$$\int_\gamma f\ dg = \int_a^b (f \circ \gamma(t)) \cdot (g \circ \gamma)'(t)\ dt$$

Then, one must check that this is well-defined up to equivalence of holomorphic differentials (if this interests you see Remark 6.12).

Remark 9. If we re-parametrize tha path (i.e., give a map $\phi : [c, d] \to [a, b]$ which is piecewise smooth) then the integral along $\gamma \circ \phi$ equals the integral along γ (checked explicitly in remark 6.13).

Example 10. If the Riemann surface S is $\mathbb{C}$, then

$$\int_\gamma f\ dg = \int_\gamma f(x)g'(x)\ dz$$

is the integral of $f(z)g'(z)$ along γ in the usual sense of complex analysis.

Example 11. If $g : S \to \mathbb{C}$ is a holomorphic mapping on any Riemann surface S then

$$\int_\gamma dg = g(\gamma(b)) = g(\gamma(a)).$$

Definition 12. If $\psi : S \to R$ is a holomorphic mapping between Riemann surfaces S and R, and if $f\ dg$ is a holomorphic differential on R then we define a holomorphic differential $\psi * (f\ dg)$ on S by

$$\psi * (f\ dg) = (f \circ \psi)d(g \circ \psi).$$

Then If $\gamma : [a, b] \to S$ is a piecewise-smooth path in S we have

$$\int_\gamma \psi * (f\ dg) = \int_a^b (f \circ \psi \circ \gamma(t))(g \circ \psi \circ \gamma)'(t)\ dt = \int_{\psi \circ \gamma} f\ dg.$$

Let C be an irreducible projective curve in $\mathbb{P}_2$ defined by a polynomial P. An **abelian integral** is an integral of the form

$$\int_\gamma f \ dg$$

where f and g are rational functions on $C - \text{sing}(C)$ and γ is a piecewise-smooth path in $C - \text{sing}(C)$ not passing through any poles of the meromorphic differential $f \ dg$.

We usually assume that C is not the line at infinity defined by $z = 0$ and take the function g to be

$$[x : y : z] \mapsto \frac{x}{z}.$$

We often then work in the affine coordinates $[x : y : 1]$ and write dx for dg. In affine coordinates f becomes a rational function $R(x, y)$ of x and y in the usual sense, and our integral is written

$$\int_\gamma f \ dg = \int_\gamma R(x, y) dx,$$

where we now regard y as a multivalued function of x via the equation $P(x, y, 1)$ which defines C in affine coordinates.

The integral is called an **elliptic integral** is C is an elliptic curve, meaning it is defined by an equation

$$y^2 = (x - \alpha_1) \cdots (x - \alpha_k),$$

where $k = 3, 4$. If k is larger then C is called a **hyperelliptic curve** and the integral $\int_y f \ dg$ is called a **hyperelliptic integral**.

Recall our lattice $\Lambda \subset \mathbb{C}$ and biholomorphism $u : \mathbb{C}/\Lambda \to C_\Lambda$. We have a meromorphic differential on C_Λ given in in homogeneous coordinates $[x, y, 1]$ by $y^{-1} \ dx$. Let

$$\eta = u * (y^{-1} \ dx).$$

Then η is a meromorphic differential on $\mathbb{C}/\Lambda$. Moreover, we will show it is holomorphic. If

$$\pi : \mathbb{C} \to \mathbb{C}/\Lambda$$

is given by $\pi(z) = \Lambda + z$ then

$$\pi * \eta = \pi * u * (y^{-1} dx) = (u \circ \pi) * (y^{-1} dx) = (p')^{-1} dp = (p')^{-1} p' dz = dz,$$

where $x : \mathbb{C} \to \mathbb{C}$ denotes the identity function, and $p(z)$ is the function used in defining u from 2 lectures ago. Since π is locally a holomorphic bijection with holomorphic inverse and dz is a holomorphic differential on $\mathbb{C}$ it follows that η has no poles, so η is a holomorphic differential on $\mathbb{C}/\Lambda$. Since u is a holomorphic bijection with holomorphic inverse it follows that $y^{-1}\, dx$ is a holomorphic differential on C_Λ.

Let $\lambda \in \Lambda$ and define a path $\gamma' : [0, 1] \to \mathbb{C}$ by $t \mapsto t\lambda$. If $\gamma = \pi \circ \gamma'$, then $\gamma(t) = \Lambda + t\lambda$, so $\gamma(0) = \gamma(1)$. Hence, γ is a piecewise smooth closed path in $\mathbb{C}/\Lambda$. Then,

$$\int_\gamma \eta = \int_{\gamma'} \pi * \eta = \int_{\gamma'} dz = \gamma'(1) - \gamma'(0) = \lambda.$$

Given any $\gamma : [a, b] \to \mathbb{C}/\Lambda$ is any piecewise smooth closed path then (by appendix lemma) we can find a continuous path $\gamma' : [a, b] \to \mathbb{C}$ so that $\pi \circ \gamma' = \gamma$ which is also piecewise smooth. Then,

$$\int_\gamma \eta = \int_{\gamma'} \pi * \eta = \int_{\gamma'} dz = \gamma'(b) - \gamma'(a) \in \Lambda.$$

We have proved:

Proposition 13.

$$\Lambda = \left\{ \int_\gamma \eta \mid \gamma \textit{ is a closed piecewise smooth path in } \mathbb{C}/\Lambda \right\}.$$

Corollary 14.

$$\Lambda = \left\{ \int_\gamma y^{-1}\, dx \mid \gamma \textit{ is a closed piecewise smooth path in } C_\Lambda \right\}.$$

Hence, we can recover the lattice from the curve C_Λ in $\mathbb{P}_2$.
Now we are also prepared to describe u^{-1}.

Proposition 15. *The inverse of* u *is given by*

$$u^{-1}(p) = \Lambda + \int_{[0:1:0]}^{p} y^{-1}\, dz$$

(note: this makes sense because if we have two paths from $[0 : 1 : 0]$ *to* p *then the difference of their integrals will be an integral over a closed path and hence an element of* Λ*.)*

6.2 Abel's Theorem

Last lecture: we saw that any complex torus $\mathbb{C}/\Lambda$ is biholomorphic to a nonsingular curve C_Λ in $\mathbb{P}^2$

Today: We study the abelian group structure. The torus $\mathbb{C}/\Lambda$ is an abelian group under addition. This gives the cubic curve C_Λ an induced abelian group structure.

This can be described completely in terms of the geometry of the curve C_Λ and it is determined by two properties:

1. The identity element is $[0 : 1 : 0]$ (the inflection point).
2. Three points p, q, r add to zero if and only if they are the three points of indersection of the cubic with a line.

Remark 16. We have already seen (Section 3) that if p_0 is an inflection point on a nonsingular projective cubic C then there is a unique additive group structure on C satisfying (1) and (2) above. Then, we can view Abel's theorem as the statement that the holomorphic bijection

$$u : \mathbb{C}/\Lambda \to C_\Lambda$$

is actually a group isomorphism with respect to this croup structure on C_Λ and the quotient group structure on $\mathbb{C}/\Lambda$.

Remark 17. Recall from earlier that a line in $\mathbb{P}_2$ meets a nonsingular cubic curve C in either (**QUIZ**):

1. 3 distinct points with multiplicity 1 (3 ordinary intersection points)
2. 2 distinct points with one having multiplicity 1 and one having multiplicity 2 (one ordinary intersection point and one tangent)
3. one point with multiplicity 3 (line is tangent at inflection point)

So for the group, this means that

1. three distinct points $p + q + r = 0$ iff p, q, r are distinct points on a line in $\mathbb{P}^2$;
2. two points $2p + q = 0$ iff the tangent to C_Λ at p passes through q;
3. and $3p = 0$ iff p is a point of inflection.

From this last comment we see that the points of order 1 or 3 are **precisely** the inflection points on C_Λ. Under the group isomorphism $u : \mathbb{C}/\Lambda \to C_\Lambda$ we see that there are exactly 9 such points in $\mathbb{C}/\Lambda$, and they can be written in the form

$$\Lambda + \frac{j}{3}\omega_1 + \frac{k}{3}\omega_2$$

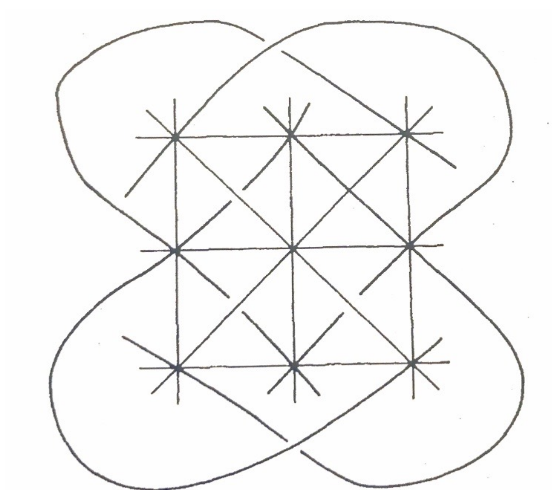
where $j, k \in \{0, 1, 2\}$.

These points form a subgroup of C/Λ isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. The three entries in any row, column, or diagonal add up to $\Lambda + 0$: It follows that C_Λ has exactly

$\Lambda + 0$	$\Lambda + \frac{1}{3}\omega_2$	$\Lambda + \frac{2}{3}\omega_2$
$\Lambda + \frac{1}{3}\omega_1$	$\Lambda + \frac{1}{3}\omega_1 + \frac{1}{3}\omega_2$	$\Lambda + \frac{1}{3}\omega_1 + \frac{2}{3}\omega_2$
$\Lambda + \frac{2}{3}\omega_1$	$\Lambda + \frac{2}{3}\omega_1 + \frac{1}{3}\omega_2$	$\Lambda + \frac{2}{3}\omega_1 + \frac{2}{3}\omega_2$

9 inflection points.

The following collections of these points on C_Λ all lie on lines in $\mathbb{P}_2$:



Recall from the end of the previous lecture that the inverse of u is given by

$$u^{-1}(p) = \Lambda + \int_{[0:1:0]}^{p} y^{-1}dx,$$

where the integra lis over any piecewise smooth path in C_Λ from $[0 : 1 : 0]$ to p.

Theorem 18 (Abel's Theorem)**.** *If* $t, v, w \in \mathbb{C}$ *then*

$$t + v + w \in \Lambda$$

if and only if there is a line L *in* $\mathbb{P}_2$ *whose intersection with* C_Λ *consists of the points* $u(\Lambda + t)$*,* $u(\Lambda + v)$*, and* $u(\Lambda + w)$ *(allowing for multiplicities).*

Equivalently, if $p, q, r \in C_\Lambda$ *then*

$$\Lambda + \int_{[0:1:0]}^{p} y^{-1} dx + \int_{[0:1:0]}^{q} y^{-1} dx + \int_{[0:1:0]}^{r} y^{-1} dx = \Lambda + 0$$

if and only if p, q, r *are the points of interesection with a line in* $\mathbb{P}_2$*.*

Remark 19. We can interpret Abel's theorem as an addition formula modulo Λ for integrals of the form

$$\int_{[0:1:0]}^{p} y^{-1} dx$$

on C_Λ.

For the rest of the talk we will prove Abel's Theorem.

Proof. First we show that if L is a line in $\mathbb{P}_2$, then

$$\Lambda + \int_{[0:1:0]}^{p} y^{-1} dx + \int_{[0:1:0]}^{q} y^{-1} dx + \int_{[0:1:0]}^{r} y^{-1} dx = \Lambda + 0$$

We do this in 3 increasingly general cases.

Case 1: Suppose L is the tangent line $z = 0$ to C_Λ at the point of inflection $[0 : 1 : 0]$. Then $p = q = r$, so the equality we wish to prove is trivial.

Case 2: Suppose that L is a line of the form $cy = bz$. Then L meets C_Λ in 3 points:

$$p_1(b, c) = [a_1, b, c]$$
$$p_2(b, c) = [a_2, b, c]$$
$$p_2(b, c) = [a_2, b, c]$$

where a_1, a_2, a_3 are the roots of the polynomial

$$Q_\Lambda(x, b, c) = b^2 c - 4x^3 + g_2(\Lambda) x c^2 + g_3(\Lambda) c^3.$$

Define a map $\mu : \mathbb{P}_1 \to \mathbb{C}/\Lambda$ by

$$\mu[b,c] = \Lambda + \int_{[0:1:0]}^{p_1(b,c)} y^{-1}dx + \int_{[0:1:0]}^{p_2(b,c)} y^{-1}dx + \int_{[0:1:0]}^{p_3(b,c)} y^{-1}dx$$

where the integrals are over any paths in $\mathbb{C}$. This map μ is well defined. We now give two lemmas.

Lemma 20. $\mu : \mathbb{P}_1 \to \mathbb{C}$ *is holomorphic.*

Lemma 21. *Any holomorphic map from* $\mathbb{P}_1$ *to* $\mathbb{C}$ *is constant.*

The proof of the first lemma will be given when the proof of Abel's theorem is complete. The second lemma is clear by homework exercises: $\mathbb{P}_1$ is compact so its image under a holomorphic map to $\mathbb{C}$ is a constant.

By the appendix Lemma there is a continuous map $\tilde{\mu} : \mathbb{P}_1 \to \mathbb{C}$ such that $\mu = \pi \circ \tilde{\mu}$. Since the inverse of the restriction of π to a suitable open neighborhood of any $a \in \mathbb{C}$ defines a holomorphic chart on a neighborhood of $\Lambda + a$ in $\mathbb{C}/\Lambda$ and μ is holomorphic, it follows that $\tilde{\mu}$ is holomorphic. By the second lemma, it is therefore a constant. Thus μ is a constant map.

By case 1, $\mu[1,0] = \Lambda + 0$, so we have that $\mu[b,c] = \Lambda + 0$ for all $[b,c] \in \mathbb{P}_1$.

Case 3: Suppose now that L is any line in $\mathbb{P}_2$. Then the equation for L can be written in the form

$$sx + t(cy - bz) = 0,$$

for some s,t not both zero and b,c both not zero. Fix b,c and define a map $\nu : \mathbb{P}_1 \to C/\Lambda$ by

$$\nu[s,t] = \Lambda + \int_{[0:1:0]}^{q_1(s,t)} y^{-1}dx + \int_{[0:1:0]}^{q_2(s,t)} y^{-1}dx + \int_{[0:1:0]}^{q_3(s,t)} y^{-1}dx$$

where $q_1(s,t), q_2(s,t), q_3(s,t)$ are the points of intersection of C_Λ with the line $sx + t(cy - bz) = 0$.

As in case 2 this map ν is holomorphic and hence constant. From case 2 we also know that $\nu[0,1] = \Lambda + 0$ so for all $[s,t] \in \mathbb{P}_1$ we have $\nu[s,t] = \Lambda + 0$. This completes one direction of the proof up to the unproven lemma.

Conversely, suppose that $t, v, w \in \mathbb{C}$ and $t + v + w \in \Lambda$. Let

$$p = u(\Lambda + t), \quad q = u(\Lambda + v), \quad r = u(\Lambda + w).$$

Let L be the line through p, q or tangent at $p = q$ if that is the case. Then, L meets C_Λ in p, q and another point $\tilde{r}$. Then by what we have just proved,

$$\begin{aligned} u^{-1}(p) + u^{-1}(q) + u^{-1}(\tilde{r}) &= \Lambda + 0 \\ &= \Lambda + t + v + w \\ &= u^{-1}(p) + u^{-1}(q) + u^{-1}(r) \end{aligned}$$

Hence $u^{-1}(r) = u^{-1}(\tilde{r})$ and so $r = \tilde{r}$. This completes the proof up to the lemma. □

Lemma 22. $\mu : \mathbb{P}_1 \to \mathbb{C}$ *is holomorphic, where*

$$\mu[b, c] = \Lambda + \int_{[0:1:0]}^{p_1(b,c)} y^{-1} dx + \int_{[0:1:0]}^{p_2(b,c)} y^{-1} dx + \int_{[0:1:0]}^{p_3(b,c)} y^{-1} dx$$

Proof. For all but finitely many $b \in \mathbb{C}$ the partial derivative

$$\frac{dQ_\lambda}{dx}(x, y, z)$$

of the polynomial Q_Λ defining C_Λ is nonzero at $(a, b, 1)$ when a is a root of the polynomial $Q_\Lambda(x, b, 1)$ in x. For such a, b, the polynomial $Q_\Lambda(x, b, 1)$ has 3 distinct roots, a_1, a_2, a_3 and

$$p_1 = [a_i, b, 1]$$

By the implicit function theorem applied to $Q_\Lambda(x, y, 1)$ there are open neighborhoods U and v_1, V_2, V_3 of b and a_1, a_2, a_3 in $\mathbb{C}$ and holomorphic functions $g_i : U \to V_i$ such that if $x \in V_i$ and $y \in U$ then

$$Q_\Lambda(x, y, 1) = 0 \;\leftrightarrow\; z = g_i(y).$$

Hence there are holomorphic maps $\psi_i : U \to C_\Lambda$ given by

$$\psi_i(w) = [g_i(w), w, 1].$$

We may choose V_1, V_2, V_3 to be disjoint. This means that if $w \in U$ then $g_1(w), g_2(w), g_3(w)$ are distinct roots of $Q_\Lambda(x, w, 1)$ and so

$$p_i(w, 1) = [g_i(w), w, 1] = \psi_i(w).$$

Thus if γ is a path in U from b to w then $\psi_i \circ \gamma$ is a path in C_Λ from $p_i(b,1)$ to $p_i(w,1)$ and

$$\int_{\psi_i \circ \gamma} y^{-1} dx = \int_\gamma \frac{g_i'(w)}{w} dw.$$

Thus

$$\mu[w,1] = \mu[b,1] + \sum_{i=1}^{3} \int_b^w \frac{g_i'(y)}{y} dy$$

where the integrals are over any path. Since $g_i'(u) = 0$ when $y = 0$ the functions $g_i'(y)/y$ are holomorphic on U so their integrals from b to w are holomorphic functions of w near b. Thus μ is holomorphic in a neighborhood of $[b,1]$. Thus we have shown that μ is holomorphic except at possibly finitely many points of $\mathbb{P}_1$. By an appendix theorem it now suffices to show that μ is continuous, which is left as an exercise. □