The Degree of SO(n)

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Goal

The goal of this project [1] was to compute the degree of the special orthogonal group as an algebraic variety.

SO(n)

The group SO(n) is defined as

$$\mathrm{SO}(n) := \mathrm{SO}(n, \mathbb{C}) = \left\{ M \in \mathrm{Mat}_{n,n}(\mathbb{C}) \mid \det M = 1, \quad M^t M = \mathrm{Id} \right\}.$$

The equations defining SO(n) are polynomials in the entries of the matrix, so that SO(n) is an algebraic variety.

Degree of an algebraic variety

The projective closure *projective closure* $\overline{X}$ of an embedded affine variety X is the smallest projective variety containing X.

The degree of a complex variety X is the maximum number of intersection points of $\overline{X}$, with a linear space L of complementary dimension. This maximum will be achieved as long as the L is chosen generically.

Sample values

This is now sequence [A280921] on OEIS [5].

n	degree of SO(n)
2	2
3	8
4	40
5	384
6	4768
7	111616
8	3433600
9	196968448
10	14994641408

Using symbolic Gröbner basis techniques, one can verify these values on a home laptop (for example, in `Macaulay2` [4]) up to $n = 5$.

Main Theorem

$$\deg \mathrm{SO}(n) = 2^{n-1} \det \left(\binom{2n-2i-2j}{n-2i} \right)_{1 \le i,j \le \lfloor \frac{n}{2} \rfloor}$$

Main ingredient for proof

The main ingredient of the proof was the following theorem of Kazarnovskii [2].

Let G be a connected reductive group of dimension m and rank r over an algebraically closed field. If $\rho : G \to \mathrm{GL}(V)$ is a representation with finite kernel then,

$$\deg \overline{\rho(G)} = \frac{m!}{|W(G)|(e_1! e_2! \cdots e_r!)^2 |\ker(\rho)|} \int_{C_V} (\check{\alpha}_1 \check{\alpha}_2 \cdots \check{\alpha}_l)^2 dv.$$

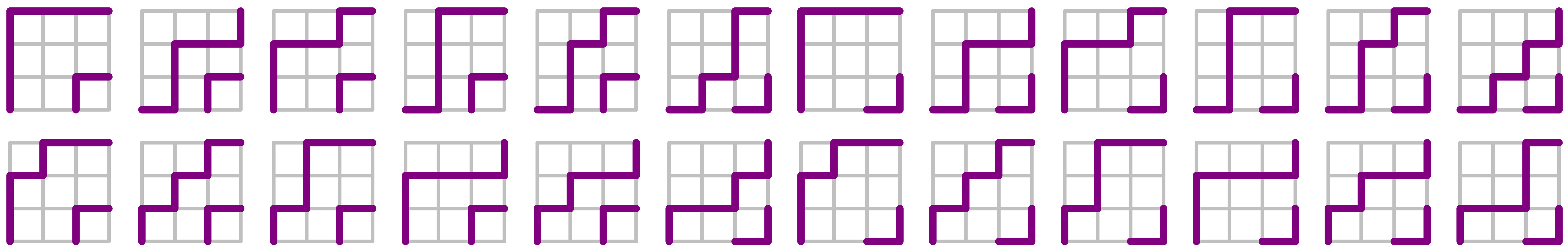
where $W(G)$ is the Weyl group, e_i are Coxeter exponents, C_V is the convex hull of the weights, and $\check{\alpha}_i$ are the coroots.

Non-Intersecting Lattice Paths

Using the Gessel-Viennot lemma [3], we can recast this result in terms of lattice paths:

$$\deg \mathrm{SO}(n) = 2^{n-1}(\#\{\text{Non-Intersecting Lattice Paths from A to B}\})$$

where the positions of A and B are given by $a_i = (2i - n, 0), b_j = (0, n - 2j)$ where $1 \le i, j \le \lfloor \frac{n}{2} \rfloor$.



The 24 non-intersecting lattice paths corresponding to deg(SO(5)).

References

[1] Madeline Brandt et al. "The degree of SO(n)". In: *Combinatorial Algebraic Geometry.* Fields Inst. Commun. 80. Fields Inst. Res. Math. Sci., 2017, pp. 207–224.
[2] Harm Derksen and Gregor Kemper. *Computational Invariant Theory.* Vol. 130. Encyclopaedia of Mathematical Sciences. Springer-Verlag, Berlin Heidelberg, 2002.
[3] Ira Gessel and Gérard Viennot. "Binomial determinants, paths, and hook length formulae". In: *Advances in mathematics* 58.3 (1985), pp. 300–321.
[4] Daniel R. Grayson and Michael E. Stillman. *Macaulay2, a software system for research in algebraic geometry.* Available at `http://www.math.uiuc.edu/Macaulay2/`.
[5] *Sequence A280921 in the Online Encyclopedia of Integer Sequences.* Available at `http://oeis.org`.

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