MATROIDS AND THEIR DRESSIANS (TALK)

MADELINE BRANDT

Abstract. In this talk we will explore Dressians of matroids. Dressians have many lives: they parametrize tropical linear spaces, their points induce regular matroid subdivisions of the matroid polytope, they parametrize valuations of a given matroid, and they are a tropical prevariety formed from certain Plücker equations. We show that initial matroids correspond to cells in regular matroid subdivisions of matroid polytopes, and we characterize matroids that do not admit any proper matroid subdivisions. An efficient algorithm for computing Dressians is presented, and its implementation is applied to a range of interesting matroids. If time permits, we will also discuss an ongoing project extending these ideas to flag matroids.

1. Intro

We begin with some notions from tropical geometry and matroid theory.

Definition 1.1. A *matroid* of rank d on n elements is a collection $\mathcal{B} \subset \binom{[n]}{d}$ called the *bases* of $\mathcal{M}$ satisfying:

(B0) $\mathcal{B}$ is nonempty,

(B1) Given any $\sigma, \sigma' \in B$ and $e \in \sigma' \backslash \sigma$, there is an element $f \in \sigma$ such that $\sigma \backslash \{f\} \cup \{e\} \in \mathcal{B}$.

A matroid $\mathcal{M}$ is called *realizable over* K if there exist vectors $v_1, \dots, v_n \in K^d$ such that the bases of K^d from these vectors are indexed by the bases of $\mathcal{M}$:

$$\mathcal{B} = \left\{ \sigma \in \binom{[n]}{d} \mid \{v_{\sigma_1}, \dots, v_{\sigma_d}\} \text{ is a basis of } K^d \right\}.$$

In this case, we write $\mathcal{M} = \mathcal{M}[v_1, \dots, v_n]$.

The *uniform matroid* $U_{d,n}$ is the matroid with basis set $\binom{[n]}{d}$.

Let K be an algebraically closed field with a valuation val_K. The *Grassmannian* $G(d, n) \subset \mathbb{P}^{\binom{n}{d}-1}$ is the image of $K^{d\times n}$ under the *Plücker embedding*, which sends a $d \times n$-matrix to the vector of its $d \times d$ minors. This vector is called the *Plücker coordinates* of the matrix. The Grassmannian is a smooth algebraic variety defined by equations called the *Plücker relations*, which give the relations among the maximal minors of the matrix. Points of this variety correspond to d-dimensional linear subspaces of K^n.

The open subset $G^0(d, n)$ of the Grassmannian parametrizes subspaces whose Plücker coordinates are all nonzero. Points in this variety correspond to equivalence classes of matrices where no minor vanishes. In other words, these are matrices which give the uniform matroid of rank d on $[n]$.

We now recall the definition of the tropical Grassmannian and Dressian of a matroid, as in [MS15]. Let $\mathcal{M}$ be a matroid of rank d on the set $E = [n]$. For any basis σ of $\mathcal{M}$, we introduce a variable p_σ. Consider the Laurent polynomial ring $K[p_\sigma^{\pm 1} \mid \sigma \text{ is a basis of } \mathcal{M}]$ in these variables.

Date: June 27, 2019.

Let $G_{\mathcal{M}}$ be the collection of polynomials obtained from the three-term Plücker relations by setting all variables not indexing a basis to zero. More precisely, these are the equations

$$G_{\mathcal{M}} = \left\{ p_{Sij}p_{Skl} - p_{Sik}p_{Sjl} + p_{Sil}p_{Sjk} \ : \ S \in \binom{n}{d-2}, i \neq j \neq k \neq l, \text{ and } p_\sigma = 0 \text{ if } \sigma \notin \mathcal{B} \right\}.$$

Let $I_{\mathcal{M}}$ be the ideal generated by $G_{\mathcal{M}}$. We call $I_{\mathcal{M}}$ the *matroid Plücker ideal* of $\mathcal{M}$, and refer to elements of $G_{\mathcal{M}}$ as *matroid Plücker relations*.

The points of the variety $V(I_{\mathcal{M}})$ correspond to realizations of the matroid $\mathcal{M}$ in the following sense. Points in $V(I_{\mathcal{M}})$ give equivalence classes of $d \times n$ matrices whose maximal minors vanish exactly when those minors are indexed by a nonbasis of $\mathcal{M}$.

Definition 1.2. We will call $V(I_{\mathcal{M}})$ the *matroid Grassmannian of* $\mathcal{M}$. The variety $V(I_{\mathcal{M}})$ is empty if and only if $\mathcal{M}$ is not realizable over K. Its tropicalization $\mathrm{Gr}_{\mathcal{M}} = \mathrm{trop}(V(I_{\mathcal{M}}))$ is called the *tropical Grassmannian of* $\mathcal{M}$.

Definition 1.3. The *Dressian* $\mathrm{Dr}_{\mathcal{M}}$ of the matroid $\mathcal{M}$ is the tropical prevariety obtained by intersecting the tropical hypersurfaces corresponding to elements of $G_{\mathcal{M}}$:

$$\mathrm{Dr}_{\mathcal{M}} = \bigcap_{f \in G_{\mathcal{M}}} \mathrm{trop}(V(f)).$$

By definition, we have that $\mathrm{Gr}_{\mathcal{M}} \subseteq \mathrm{Dr}_{\mathcal{M}}$, and equality holds if and only if the matroid Plücker relations form a tropical basis.

2. Matroid Polytopes and Valuations

The *matroid polytope* $P_{\mathcal{M}}$ of $\mathcal{M}$ is the convex hull of the indicator vectors of the bases of $\mathcal{M}$:

$$P_{\mathcal{M}} = \mathrm{conv}\{e_{\sigma_1} + \cdots + e_{\sigma_d} \mid \sigma \in \mathcal{B}\}.$$

The dimension of $P_{\mathcal{M}}$ is $n - c$, where c is the number of connected components of $\mathcal{M}$ [FS05].

The following celebrated result gives a simple way to check whether or not a polytope is a matroid polytope.

Theorem 2.1 (GGMS Theorem, 4.2.12 in [MS15])**.** *A polytope* P *with vertices in* $\{0,1\}^{n+1}$ *is a matroid polytope if and only if every edge of* P *is parallel to* $e_i - e_j$*.*

Points in the Dressian of $\mathcal{M}$ have an interesting relationship to the matroid polytope of $\mathcal{M}$. Every vector w in $\mathbb{R}^{|B|}/\mathbb{R}1$ induces a regular subdivision Δ_w of the polytope $P_{\mathcal{M}}$. A subdivision of the matroid polytope $P_{\mathcal{M}}$ is a *matroid subdivision* if all of its edges are translates of $e_i - e_j$. Equivalently, by Theorem 2.1, this implies all of the cells of the subdivision are matroid polytopes.

Proposition 2.2 (Lemma 4.4.6, [MS15])**.** *Let* $\mathcal{M}$ *be a matroid, and let* $w \in \mathbb{R}^{|B|}$*. Then* w *lies in the Dressian* $\mathrm{Dr}_{\mathcal{M}}$ *if and only if the corresponding regular subdivision* Δ_w *of* $P_{\mathcal{M}}$ *is a matroid subdivision.*

Question 2.3. What is the relationship of these smaller matroids appearing in regular matroid subdivisions of the matroid polytope to the original matroid $\mathcal{M}$?

The *lineality space* of a tropical (pre)variety T is the largest linear space L such that for any point $w \in T$ and any point $v \in L$, we have that $w + v \in T$.

All matroids admit the trivial subdivision of their matroid polytope as a regular matroid subdivision, so the Dressian $\mathrm{Dr}_{\mathcal{M}}$ is nonempty for all matroids $\mathcal{M}$. This gives us the lineality space of the Dressian, as we see in the following proposition.

Proposition 2.4 ([OPS18] , [DW92]). *Let $\mathcal{M}$ be a matroid, and let c be the number of connected components of $\mathcal{M}$. The lineality space of $\mathrm{Dr}_{\mathcal{M}}$ has dimension $n - c$ (in $\mathbb{R}^{|\mathcal{B}|}/\mathbb{R}1$) and is given by the image of the map $\mathbb{R}^n \to \mathbb{R}^{|\mathcal{B}|}$ given by $e_i \mapsto \sum_{B \ni i} e_B$.*

We now discuss valuated matroids, as in [DW92].

Definition 2.5. Let $\mathcal{M}$ be a matroid on $E = \{1, \dots, n\}$ of rank d and bases $\mathcal{B}$. Let $\nu : \mathcal{B} \to \mathbb{R} \cup \infty$ be a vector so that the pair (M, ν) satisfies the following version of the exchange axiom:

(V0) for $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \backslash B_2$, there exists an $f \in B_2 \backslash B_1$ with $B_1' = (B_1 \backslash \{e\}) \cup \{f\} \in \mathcal{B}$, $B_2' = (B_2 \backslash \{f\}) \cup \{e\} \in \mathcal{B}$, and $\nu(B_1) + \nu(B_2) \geq \nu(B_1') + \nu(B_2')$.

We will call ν a *valuation on* $\mathcal{M}$, and the pair $(\mathcal{M}, \nu)$ is called a *valuated matroid* (See [DW92] for details).

It is known that valuations on a matroid $\mathcal{M}$ are exactly the points in $\mathrm{Dr}_{\mathcal{M}}$ [MS15]. Indeed, the above condition asserts exactly that the tropicalized matroid Plücker relations hold.

Definition 2.6. Let $\mathcal{M}$ be a matroid with bases $\mathcal{B}$ and let $\nu \in \mathrm{Dr}_{\mathcal{M}}$. Then the *initial matroid* $\mathcal{M}_\nu$ is the matroid whose bases are $\mathcal{B}_\nu = \{\sigma \in \mathcal{B} \mid \nu(\sigma) \text{ is minimal}\}$. Given a matroid $\mathcal{M}$, the *initial matroids* of $\mathcal{M}$ are the matroids $\mathcal{M}'$ such that there exists a $\nu \in \mathrm{Dr}_{\mathcal{M}}$ with $\mathcal{M}_\nu = \mathcal{M}'$.

Theorem 2.7. *Let $\mathcal{M}$ be a matroid with matroid polytope $P_{\mathcal{M}}$, let ν be a valuation on $\mathcal{M}$, let L be the lineality space of the Dressian of $\mathcal{M}$, and let Δ_ν be the matroid subdivision of $P_{\mathcal{M}}$ induced by ν. Then,*

$$\Delta_\nu = \{P(\mathcal{M}_w) \mid w \in \nu + L\}.$$

Example 2.8. Let $\mathcal{M} = U_{2,4}$, the uniform rank 2 matroid on 4 elements; $\mathcal{B} = \{01, 02, 03, 12, 13, 23\}$.

We now study the Dressian of $\mathcal{M}$. In this case, $G_{\mathcal{M}} \subset \mathbb{C}[p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}]$ consists of the single equation $p_{03}p_{12} - p_{02}p_{13} + p_{01}p_{23}$. So, we have that the Dressian Dr_M and the Grassmannian $\mathrm{Gr}_{\mathcal{M}}$ coincide, and they are both described by

$$\min\{p_{03} + p_{12}, p_{02} + p_{13}, p_{01} + p_{23}\} \text{ is attained twice.}$$

The Dressian is a 5 dimensional fan with a four dimensional lineality space. Let the basis for $\mathbb{R}^{|\mathcal{B}|}$ be given by $\{e_{01}, e_{02}, e_{03}, e_{12}, e_{13}, e_{23}\}$. Then, The lineality space is given by

$$L = \mathrm{span}\left((1, 1, 1, 0, 0, 0), (1, 0, 0, 1, 1, 0), (0, 1, 0, 1, 0, 1), (0, 0, 1, 0, 1, 1)\right).$$

The Dressian $\mathrm{Dr}_{\mathcal{M}}$ has 3 maximal cones which are each generated by a ray. The rays are spanned by the points

$$r_{01,23} = (1, 0, 0, 0, 0, 1) \quad r_{02,13} = (0, 1, 0, 0, 1, 0) \quad r_{03,12} = (0, 0, 1, 1, 0, 0).$$

The matroid polytope $P_{\mathcal{M}}$ is the hypersimplex $\Delta(2, 4)$, which is an octahedron. Each of the cones of $\mathrm{Dr}_{\mathcal{M}}$ corresponds to a subdivision of $P_{\mathcal{M}}$ in to two pyramids. Let us study points in the cell of $\mathrm{Gr}_{\mathcal{M}}$ containing $r_{01,23}$. The point $r_{01,23}$ induces a subdivision where the two maximal cells are the pyramids which are the convex hulls of

$$P_{01} = \mathrm{conv}\{e_{01}, e_{02}, e_{03}, e_{12}, e_{13}\}, \quad P_{23} = \mathrm{conv}\{e_{23}, e_{02}, e_{03}, e_{12}, e_{13}\}.$$

The matroid $\mathcal{M}_{r_{01,23}}$ has bases $\{02, 03, 13, 12\}$. Its matroid polytope is the square face which is shared by they pyramids p_{01} and p_{23}. Over $\mathbb{C}\{\{t\}\}$, we can realize $\mathcal{M}$ with the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1+t & 1+2t & t & 2t \end{bmatrix},$$

and the resulting Plücker vector valuates to $r_{01,23}$. This matrix reduces to a matrix over $\mathbb{C}$ whose matroid is $\mathcal{M}_{r_{01,23}}$. Alternatively, we can also realize $\mathcal{M}$ with the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 3+t^2 \end{bmatrix}.$$

The Plücker coordinate of this matrix valuates to

$$\nu = (0,0,0,0,0,2) = r_{01,23} - (1,0,0,0,0,-1) \in r_{01,23} + L.$$

The matroid $\mathcal{M}_\nu$ is the matroid with bases $\{01, 02, 03, 12, 13\}$, whose matroid polytope is p_{01}. Additionally, the matrix above reduces to a matrix over $\mathbb{C}$ whose matroid is exactly $\mathcal{M}_\nu$.

3. Rigidity

We say a matroid is *rigid* if the only regular matroid subdivision of the matroid polytope is the trivial subdivision.

Proposition 3.1 ([OPS18],[DW92])**.** *All binary matroids are rigid* [OPS18, DW92] *and every finite projective space of dimension at least two is rigid* [DW92].

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be matroids with disjoint ground sets E_1 and E_2 respectively, and basis sets $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. The *direct sum* of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the matroid $\mathcal{M}_1 \oplus \mathcal{M}_2$ with ground set $E_1 \cup E_2$ and bases $B_1 \cup B_2$ such that $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$. A matroid is *connected* if it cannot be written as the direct sum of other matroids. The number of connected components of a matroid is the number of connected matroids it is a direct sum of.

Proposition 3.2. *Let $\mathcal{M}$ be a rank d matroid on n elements. Then $\mathcal{M}$ is rigid if and only if every matroid $\mathcal{M}'$ different from $\mathcal{M}$ with $\mathcal{M}' \prec \mathcal{M}$ has more components than $\mathcal{M}$.*

4. Computation

Using software (for instance, `Gfan` [Jen]), we may compute tropical prevarieties and varieties. However, these computations become unfeasible for inputs with many equations or variables. We can give a reduction algorithm for matroid Plücker relations, which we use in the computations in the remainder of the paper. In the described coordinates, the Dressian $\mathrm{Dr}_\mathcal{M}$ of a matroid $\mathcal{M}$ will have a large *linearity space* and *lineality space* (see Figure 1). The *linearity space* is the affine span of $\mathrm{Gr}_\mathcal{M}$.

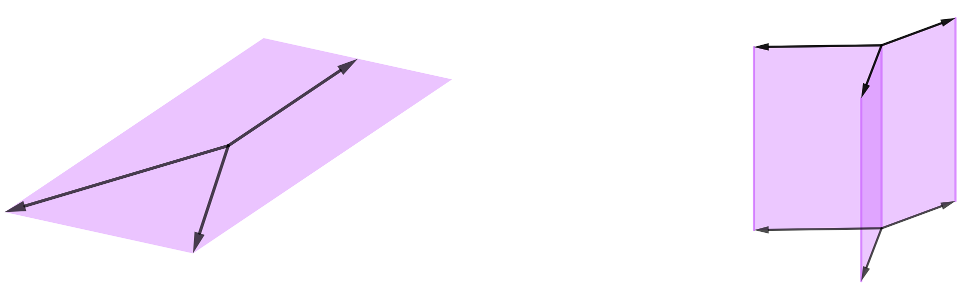

Figure 1. Linearity and Lineality

If the linearity space is a proper affine subspace of $\mathbb{R}^{|\mathcal{B}|}$, then the algorithm can be used to reduce the number of variables and equations by giving equations whose prevariety is equivalent via projection onto the linearity space. The generators $G_{\mathcal{M}}$ described above will typically have many binomials because they are obtained from trinomials by setting some of the variables to 0. Binomials introduce linearity into $\mathrm{Gr}_{\mathcal{M}}$:

$$xy - zw = 0 \quad \xrightarrow{\text{trop}} \quad \min x + y, z + w \text{ achieved twice } \leftrightarrow x + y = z + w.$$

So, the basic idea of the algorithm is to substitute $x \mapsto zw/y$ in all equations (one has to be a bit careful in doing this).

Consider the rank 3 matroid $\mathcal{M}_\star$ on $\{0, 1, \ldots, 9\}$ with nonbases given by the figure.

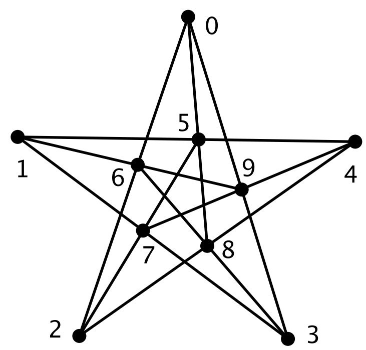

FIGURE 2. The star matroid $\mathcal{M}_\star$.

Proposition 4.1. *Modulo lineality and intersecting with a sphere, the Dressian $\mathrm{Dr}_{\mathcal{M}_\star}$ is a 2 dimensional polyhedral complex with 30 vertices, 65 edges, and 20 triangles. It is depicted in Figure* **??**. *In characteristic 0, the Grassmannian $\mathrm{Gr}_{\mathcal{M}_\star}$ is a graph with 30 vertices and 55 edges. It is depicted in Figure* **??** *in the darker color.*

We take the generators $G_{\mathcal{M}_\star}$ and make a new generating set whose tropical prevariety will not have linearity. Initially, we are working with 1260 equations in 110 unknowns. After applying the reduction algorithm, we have 73 equations in 17 unknowns.

Present work: Extending the ideas in this talk to flag matroids, with Chris Eur, Leon Zhang, and Matt Baker.

REFERENCES

[DW92] Andreas W. M. Dress and Walter Wenzel. Valuated matroids. *Adv. Math.*, 93(2):214–250, 1992.

[FS05] Eva Maria Feichtner and Bernd Sturmfels. Matroid polytopes, nested sets and Bergman fans. *Port. Math. (N.S.)*, 62(4):437–468, 2005.

[Jen] Anders N. Jensen. Gfan, a software system for Gröbner fans and tropical varieties. Available at http://home.imf.au.dk/jensen/software/gfan/gfan.html.

[MS15] Diane Maclagan and Bernd Sturmfels. *Introduction to Tropical Geometry:*. Graduate Studies in Mathematics. American Mathematical Society, 2015.

[OPS18] Jorge Alberto Olarte, Marta Panizzut, and Benjamin Schröter. On local dressians of matroids, 2018.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, 970 EVANS HALL, BERKELEY, CA 94720
E-mail address: brandtm@berkeley.edu