Slack Realization Spaces of Matroids

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Goal: Make algebraic varieties whose points correspond to realizations of a matroid.

Why: Once we have an albebraic variety describing realizations of a matroid, we can perform computations on the space, and these computations can quickly answer questions about the matroid. For example, we study the questions of: Realizability, Projective Uniqueness

1 Matroids

Catchphrase: matroids are objects which give a combinatorial abstraction of linear independence in vector spaces.

Definition 1. A **rank $d+1$ Matroid on n elements** is a subset $\mathcal{B}$ of $\binom{\{1,\ldots,n\}}{d+1}$ called the **bases** of the matroid, satisfying:

- $\mathcal{B}$ is nonempty,
- If $A, B \in \mathcal{B}$ and $a \in A \backslash B$ then there exists $b \in B \backslash A$ such that $A \backslash \{a\} \cup \{b\} \in \mathcal{B}$.

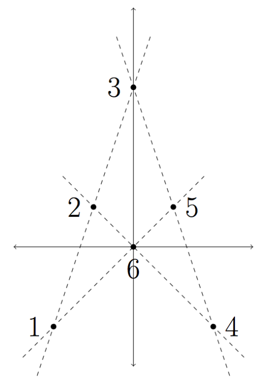
Example 2 (Realizabele Matroids)**.** Given a vector space V over a field k and vectors $v_1, \ldots, v_n \in V$ spanning V, the collection of subsets of $\{1, \ldots, n\}$ indexing bases of V gives a matroid which we denote $M[V]$. Such a matroid is called **realizable over k**, and $v_1, \ldots, v_n$ are called a **realization**.

There are examples of matroids which are not realizable. The question of whether or not a given matroid is realizable depends very much on the choice of field.

Definition 3. Let $M = M[V]$ be a realizable matroid with realization V. The **hyperplanes** of the matroid are collections of the $v_1, \dots, v_n$ which are contained in a subspace of dimension d.

Example 4. Consider the rank 3 matroid $M[V]$ for V whose vectors are

$$\begin{aligned} v_1 &= (-2,-2,1), & v_2 &= (-1,1,1), \\ v_3 &= (0,4,1), & v_4 &= (2,-2,1), \\ v_5 &= (1,1,1), & v_6 &= (0,0,1). \end{aligned}$$



Projecting onto the plane $z = 1$, this can be visualized as the points of intersection of four lines in the plane. The hyperplanes of this matroid are the four lines, together with the lines joining $2 \leftrightarrow 5$, $3 \leftrightarrow 6$, and $1 \leftrightarrow 4$.

2 Slack Matrices

Definition 5. The **slack matrix** of the matroid $M = M[V]$ over k is the $n \times h$ matrix $S_M = V^\top W$, where

- W is the matrix whose columns are the hyperplane defining normals,
- V is the matrix with columns $v_1, \dots, v_n$.

Example 6. Consdier the matroid from the previous example. Using the realization discussed there, we can compute a slack matrix for this matorid as follows.

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} -2 & -2 & 1 \\ -1 & 1 & 1 \\ 0 & 4 & 1 \\ 2 & -2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{c} \begin{array}{ccccccc} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 \\ \scriptstyle 123 & \scriptstyle 246 & \scriptstyle 345 & \scriptstyle 156 & \scriptstyle 25 & \scriptstyle 14 & \scriptstyle 36 \end{array} \\ \begin{bmatrix} -3 & 3 & 6 & -3 & 0 & 0 & 4 \\ 1 & 3 & 2 & 3 & 2 & 4 & 0 \\ -4 & 0 & -8 & 0 & -2 & 8 & 0 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{ccccccc} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 \\ \scriptstyle 123 & \scriptstyle 246 & \scriptstyle 345 & \scriptstyle 156 & \scriptstyle 25 & \scriptstyle 14 & \scriptstyle 36 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} 0 & -12 & -24 & 0 & -6 & 0 & -8 \\ 0 & 0 & -12 & 6 & 0 & 12 & -4 \\ 0 & 12 & 0 & 12 & 6 & 24 & 0 \\ -12 & 0 & 0 & -12 & -6 & 0 & 8 \\ -6 & 6 & 0 & 0 & 0 & 12 & 4 \\ -4 & 0 & -8 & 0 & -2 & 8 & 0 \end{bmatrix} \end{array}$$

In particular, we form and zero pattern of the matrix. The rows correspond to points of the matroid and the columns correspond to hyperplanes of the matroid. There is a zero in the matrix whenever a poitn is contained in a hyperplane, and a nonzero entry otherwise. We think of this value as recording the "slack" of that point with respect to that hyperplane.

Lemma 7. *The rows of a slack matrix S_M form a realization of the matroid M.*

Theorem 8 (B-Wiebe)**.** *A matrix $S \in k^{n\times h}$ is the slack matrix of some realization of M if and only if both of the following hold:*

1. *supp*$(S) =$ *supp*$(S_{M[V]})$
2. *rank*$(S) = d + 1$.

We note that these are algebraic conditions on the entries of the matrix. Given a matrix whose entries are distinct varibles, the first part of the theorem asserts that some of the variables are equal to zero. The second part of the theroem asserts that all $(d + 2) \times (d + 2)$ minors of the matrix vanish.

3 Slack Ideal

The **symbolic slack matrix** of matroid M is the matrix $S_M(\mathbf{x})$ with rows indexed by elements $i \in E$, columns indexed by hyperplanes $H_j \in \mathcal{H}(M)$ and (i, j)-entry

$$\begin{cases} x_{ij} & \text{if } i \notin H_j \\ 0 & \text{if } i \in H_j. \end{cases}$$

The **slack ideal** of M is the saturation of the ideal generated by the $(d+2)$-minors of $S_M(\mathbf{x})$, namely

$$I_M := \Big\langle (d + 2) - \text{minors of } S_M(\mathbf{x}) \Big\rangle : \left(\prod_{i=1}^{n} \prod_{j:i\notin H_j} x_{ij} \right)^{\infty} \qquad \subset k[\mathbf{x}].$$

Theorem 9 (B-Wiebe)**.** *Let M be a rank $d + 1$ matroid. Then V is a realization of M if and only if $S_{M[V]} \in \mathcal{V}(I_M) \cap (k^*)^t$.*

Example 10. Consider again the example from before. Its symbolic slack matrix is:

$$
\begin{array}{c|ccccccc}
 & H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 \\
 & 123 & 246 & 345 & 156 & 25 & 14 & 36 \\
\hline
1 & 0 & x_{12} & x_{13} & 0 & x_{15} & 0 & x_{17} \\
2 & 0 & 0 & x_{23} & x_{24} & 0 & x_{26} & x_{27} \\
3 & 0 & x_{32} & 0 & x_{34} & x_{35} & x_{36} & 0 \\
4 & x_{41} & 0 & 0 & x_{44} & x_{45} & 0 & x_{47} \\
5 & x_{51} & x_{52} & 0 & 0 & 0 & x_{56} & x_{57} \\
6 & x_{61} & 0 & x_{63} & 0 & x_{65} & x_{66} & 0
\end{array}
$$

Now, we take the 4×4 minors and saturate to obtain its slack ideal. There are 72 binomial generators of its slack ideal:

deg 2	$x_{36}x_{65} + x_{35}x_{66},\ x_{26}x_{63} - x_{23}x_{66},\ x_{15}x_{63} - x_{13}x_{65},\ x_{56}x_{61} - x_{51}x_{66},\ x_{45}x_{61} - x_{41}x_{65},$ $x_{27}x_{56} + x_{26}x_{57},\ x_{36}x_{52} - x_{32}x_{56},\ x_{17}x_{52} - x_{12}x_{57},\ x_{47}x_{51} - x_{41}x_{57},\ x_{17}x_{45} + x_{15}x_{47},$ $x_{35}x_{44} - x_{34}x_{45},\ x_{27}x_{44} - x_{24}x_{47},\ x_{26}x_{34} - x_{24}x_{36},\ x_{15}x_{32} - x_{12}x_{35},\ x_{17}x_{23} - x_{13}x_{27}$
deg 3	$x_{47}x_{56}x_{65} - x_{45}x_{57}x_{66},\ x_{17}x_{56}x_{65} + x_{15}x_{57}x_{66},\ x_{12}x_{56}x_{65} + x_{15}x_{52}x_{66},\ x_{26}x_{47}x_{65} + x_{27}x_{45}x_{66},$ $x_{26}x_{44}x_{65} + x_{24}x_{45}x_{66},\ x_{17}x_{26}x_{65} - x_{15}x_{27}x_{66},\ x_{17}x_{56}x_{63} + x_{13}x_{57}x_{66},\ x_{12}x_{56}x_{63} + x_{13}x_{52}x_{66},$ $x_{27}x_{45}x_{63} + x_{23}x_{47}x_{65},\ x_{24}x_{45}x_{63} + x_{23}x_{44}x_{65},\ x_{12}x_{36}x_{63} + x_{13}x_{32}x_{66},\ x_{24}x_{35}x_{63} + x_{23}x_{34}x_{65},$ $x_{23}x_{57}x_{61} + x_{27}x_{51}x_{63},\ x_{15}x_{57}x_{61} + x_{17}x_{51}x_{65},\ x_{13}x_{57}x_{61} + x_{17}x_{51}x_{63},\ x_{35}x_{52}x_{61} + x_{32}x_{51}x_{65},$ $x_{15}x_{52}x_{61} + x_{12}x_{51}x_{65},\ x_{13}x_{52}x_{61} + x_{12}x_{51}x_{63},\ x_{26}x_{47}x_{61} + x_{27}x_{41}x_{66},\ x_{23}x_{47}x_{61} + x_{27}x_{41}x_{63},$ $x_{13}x_{47}x_{61} + x_{17}x_{41}x_{63},\ x_{36}x_{44}x_{61} + x_{34}x_{41}x_{66},\ x_{26}x_{44}x_{61} + x_{24}x_{41}x_{66},\ x_{23}x_{44}x_{61} + x_{24}x_{41}x_{63},$ $x_{35}x_{47}x_{56} + x_{36}x_{45}x_{57},\ x_{34}x_{47}x_{56} + x_{36}x_{44}x_{57},\ x_{17}x_{35}x_{56} - x_{15}x_{36}x_{57},\ x_{35}x_{47}x_{52} + x_{32}x_{45}x_{57},$ $x_{34}x_{47}x_{52} + x_{32}x_{44}x_{57},\ x_{27}x_{34}x_{52} + x_{24}x_{32}x_{57},\ x_{13}x_{26}x_{52} + x_{12}x_{23}x_{56},\ x_{36}x_{45}x_{51} + x_{35}x_{41}x_{56},$ $x_{32}x_{45}x_{51} + x_{35}x_{41}x_{52},\ x_{12}x_{45}x_{51} + x_{15}x_{41}x_{52},\ x_{36}x_{44}x_{51} + x_{34}x_{41}x_{56},\ x_{32}x_{44}x_{51} + x_{34}x_{41}x_{52},$ $x_{26}x_{44}x_{51} + x_{24}x_{41}x_{56},\ x_{27}x_{36}x_{45} - x_{26}x_{35}x_{47},\ x_{17}x_{32}x_{44} + x_{12}x_{34}x_{47},\ x_{15}x_{23}x_{44} + x_{13}x_{24}x_{45},$ $x_{17}x_{26}x_{35} + x_{15}x_{27}x_{36},\ x_{13}x_{26}x_{35} + x_{15}x_{23}x_{36},\ x_{15}x_{27}x_{34} + x_{17}x_{24}x_{35},\ x_{15}x_{23}x_{34} + x_{13}x_{24}x_{35},$ $x_{17}x_{26}x_{32} + x_{12}x_{27}x_{36},\ x_{13}x_{26}x_{32} + x_{12}x_{23}x_{36},\ x_{17}x_{24}x_{32} + x_{12}x_{27}x_{34},\ x_{13}x_{24}x_{32} + x_{12}x_{23}x_{34}$
deg 4	$x_{27}x_{35}x_{52}x_{63} - x_{23}x_{32}x_{57}x_{65},\ x_{17}x_{36}x_{44}x_{63} - x_{13}x_{34}x_{47}x_{66},\ x_{24}x_{35}x_{57}x_{61} - x_{27}x_{34}x_{51}x_{65},$ $x_{23}x_{34}x_{52}x_{61} - x_{24}x_{32}x_{51}x_{63},\ x_{12}x_{36}x_{47}x_{61} - x_{17}x_{32}x_{41}x_{66},\ x_{13}x_{32}x_{44}x_{61} - x_{12}x_{34}x_{41}x_{63},$ $x_{15}x_{26}x_{44}x_{52} - x_{12}x_{24}x_{45}x_{56},\ x_{13}x_{26}x_{45}x_{51} - x_{15}x_{23}x_{41}x_{56},\ x_{12}x_{23}x_{44}x_{51} - x_{13}x_{24}x_{41}x_{52}$

4 Projectively Unique Matroids

We say two realizations V and V' of a matroid M are projectively equivalent if $V' = AVB$ for some $A \in GL(k^{d+1})$ and B is a k^*-multiple of a permutation matrix.

Lemma 11. *Two realizations of a matroid M are projectively equivalent if and only if their slack matrices are the same up to row and column scaling.*

Proposition 12. *The slack variety is closed under the action of the group $T_{n,h}$, where $(k^*)^n$ acts by row scaling and $(k^*)^h$ acts by column scaling.*

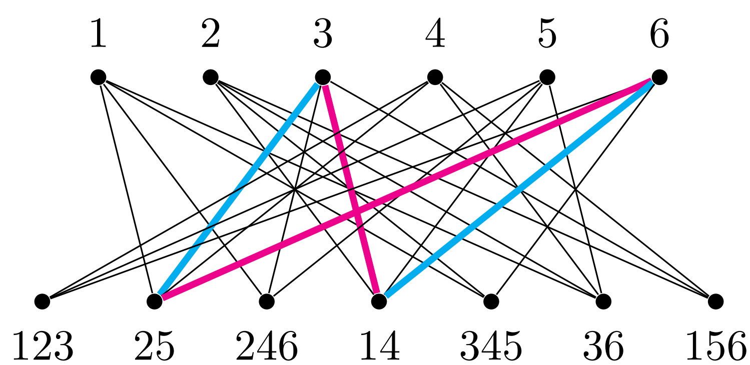
When a matroid is **projectively unique**, there is a single realization up to projective transformations; in other words, $\mathcal{V}(I_M)$ is the toric variety which is the closure of the orbit of some realization under the action of $T_{n,h}$. This implies $\sqrt{I_M} = \mathcal{I}(\mathcal{V}(I_M))$ is a toric ideal when M is projectively unique.

Example 13. The matroid from the previous examples is projectively unique.

Question 14. When is a slack ideal toric?

Definition 15. Define the *non-incidence graph of matroid* M as the bipartite graph G_M with one node for each element of the ground set of M, one node for each hyperplane, and an edge between element i and hyperplane H_j if and only if $i \notin H_j$.

Example 16. The graph G_{M_4} for matroid M_4 with the highlighted cycle corresponding to the binomial $x_{36}x_{65} + x_{35}x_{66}$.



Definition 17. Let M be an abstract matroid with realization V. Let $s \in (k^*)^t$ be such that $s = S_{M[V]}$. We define the **cycle ideal** C_V of M[V] to be the ideal

$$C_V = \left\langle \mathbf{x}^{c+} - \alpha_c \mathbf{x}^{c-} : c \text{ is a cycle in } G_M \text{ and } \alpha_c = \frac{s^{c+}}{s^{c-}} \right\rangle \subseteq k[x] \tag{1}$$

where c+ and c− are alternating edges from the cycle c.

Theorem 18. *The ideal C_V is the (scaled) toric ideal which is the kernel of the* k*-algebra homomorphism $\phi : k[x] \to k[r, t, r^{-1}, t^{-1}]$, which sends $x_{ij} \mapsto s_{i,j} r_i t_j$.*

Theorem 19. *If the slack ideal of a matroid is cyclic (i.e. equals C_V) then* M *is projectively unique and I_M is radical. The converse also holds when* k *is algebraically closed.*