A tropical count of binodal cubic surfaces

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Tropical Methods in Real Algebraic Geometry

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An Enumerative Problem

Given

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- a degree d
- a dimension n
- a number of nodes k

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We may ask: how many complex k-nodal degree d hypersurfaces in $\mathbb{P}^n_{\mathbb{C}}$ are there passing through the m points?

For correct values of m, d, n and k this is a finite number.

Example: Cubic Surfaces

Parametrize cubic surfaces X in $\mathbb{P}^3$ by their coefficients:

$$X = V(f),$$

$$f(x, y, z, w) = a_{300}x^3 + a_{210}x^2y + \cdots + a_{000}w^3.$$

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If I fix a point $[x_0 : y_0 : z_0 : w_0] \in \mathbb{P}^3$, the collection of all cubic surfaces passing through that point forms a hyperplane in $\mathbb{P}^{19}$.

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$$\begin{array}{lcll} \text{the degree of the} & & \text{the number of singular cubic surfaces} & \\ & = & & = \quad 32 \\ \text{discriminant} & & \text{passing through 18 generic points} & \end{array}$$

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the degree of the discriminant	$=$	the number of singular cubic surfaces passing through 18 generic points	$=$	32

The singular locus of the discriminant is reducible:

$$\operatorname{sing}(\Delta) = \{\text{binodal cubic surfaces}\} \cup \{\text{cuspidal cubic surfaces}\}$$

Each of these is a codimension 2 variety in $\mathbb{P}^{19}$.

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Theorem (Vainsencher)

There are $2(d-2)(4d^3-8d^2+8d-25)(d-1)^2$ complex binodal degree d surfaces in $\mathbb{P}^3$ through $\binom{d+3}{3}-3$ points in general position.

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Setting $d=3$, we see that there **280 binodal cubic surfaces** through 17 points in general position.

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Theorem (B-Geiger)

There are 39 ***tropical binodal cubic surfaces*** *through 17 points in* ***Mikhalkin's position*** *containing* ***separated singularities****. They give rise to 214 complex binodal cubic surfaces through 17 points.*

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$$280 = \sum_{S} \mathrm{mult}(S),$$

where the sum is over all binodal tropical cubic surfaces through the tropicalizations of the 17 points, and $\mathrm{mult}(S)$ is the number of classical binodal cubics tropicalizing to S.

If we count all tropical binodal cubic surfaces through our points with multiplicities, we will recover the true count.

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Definition

Points in $\mathbb{R}^3$ are in **Mikhalkin position** if they are distributed with growing distances on a line $\{\lambda \cdot (1, \eta, \eta^2) | \lambda \in \mathbb{R}\} \subset \mathbb{R}^3$, where $0 < \eta \ll 1$.

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Tropical surfaces through these points have well-understood Newton subdivisions (Markwig-Markwig-Shustin).

This allows us to study only 39 subdivisions of the Newton polytope of a cubic surface (compared to 344,843,867).

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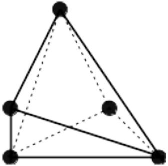
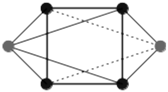
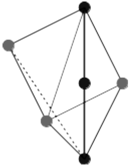
For us, there are only a few types of singularities that can occur. They are in the portion of the tropical surface dual to the following figures which could appear in the Newton subdivision (Markwig-Markwig-Shustin):

Separated nodes

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As we count, we do encounter some cases with unseparated nodes. At this time, we do not understand whether these surfaces can actually occur and with what multiplicity to count them.

This is why we only have 214 surfaces in our count instead of 280. The remaining 66 tropical surfaces all have unseparated nodes.

Overview of strategy

We follow (Markwig-Markwig-Shustin-Shaw).

Lattice path: Each point in the point configuration is contained in the interior of a 2-dimensional cell. Encode this in the Newton polytope by a lattice path.

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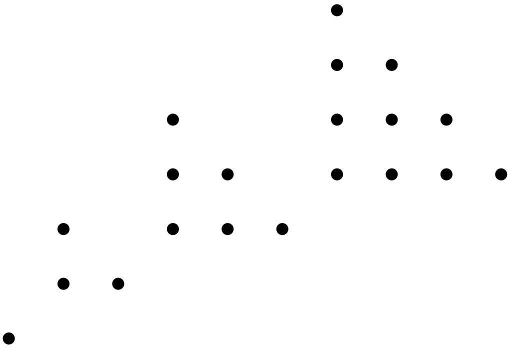
Multiplicities: Count with multiplicity.

Example



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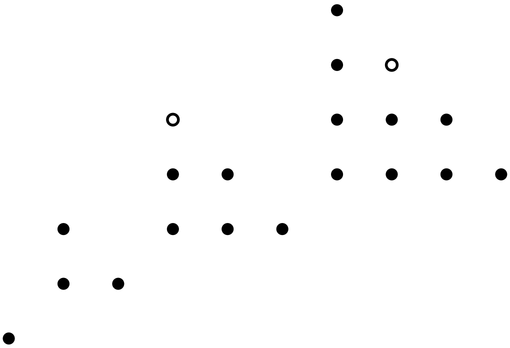
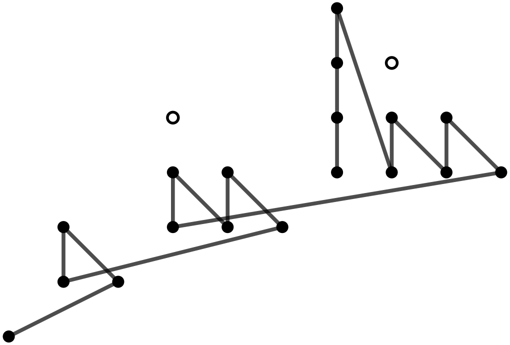
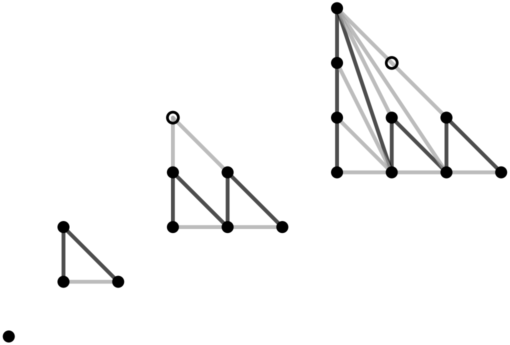


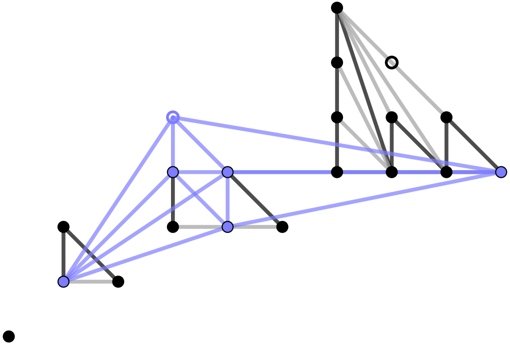
Example: lattice path



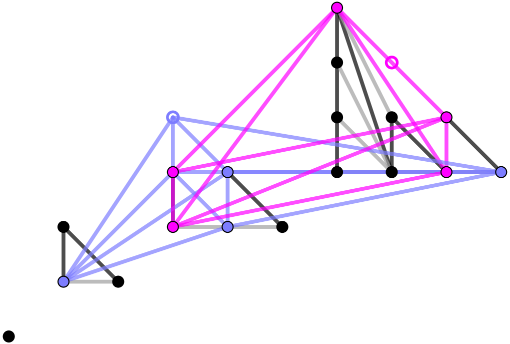
Example: floor plan



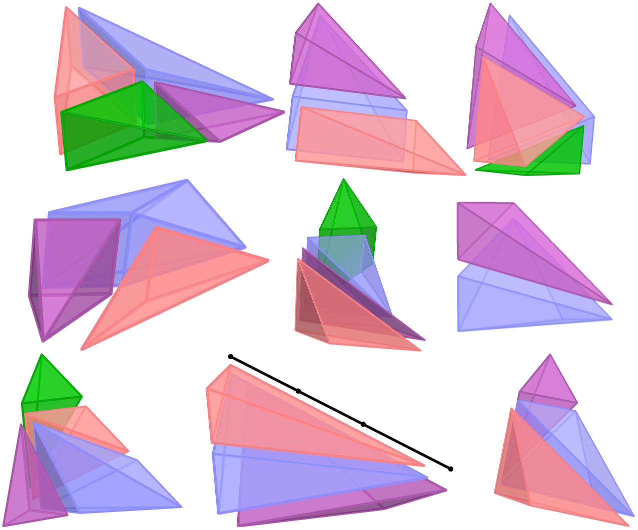
Example: bipyramid



Example: count with multiplicity 8



Complexes that could be dual to two nodes



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Thank You