Intersecting Hypergraphs and Decompositions of Complete Uniform Hypergraphs

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Abstract

This thesis explores the relationship between two theorems from extremal combinatorics: the Erdős-Ko-Rado theorem and the Baranyai theorem. Seven proofs of the Erdős-Ko-Rado theorem are given, and the proof of the Baranyai theorem is also presented. We provide a discussion of the wreath conjecture, an extension of the Baranyai theorem, and subsequently give a proof that the wreath conjecture implies the Erdős-Ko-Rado theorem.

Introduction

Extremal combinatorics is a field of mathematics that addresses questions of the size of finite objects given that the objects satisfy certain constraints. Hypergraphs, an object of study in extremal combinatorics, are a generalization of graphs where an edge can contain any number of vertices. In this thesis, we will explore the relationship between two theorems about uniform hypergraphs (hypergraphs in which all edges have the same size).

The first theorem is the Erdős-Ko-Rado theorem, which bounds the size of an intersecting hypergraph. As an example, suppose that Europe has twenty-four languages, and you wish to assemble a group of diplomats such that each diplomat in the group speaks exactly four languages, and any two diplomats speak a common language. What is the maximum number of diplomats?

The second theorem is the Baranyai theorem, which guarantees the existence of a certain decomposition of a complete uniform hypergraph. As an example, suppose now that we have a group of nine diplomats such that any set of three diplomats can communicate using one of eighty-four total languages, and only three diplomats speak every language. The diplomats are attending a conference which lasts all twenty-eight days of February. Each day at lunch, the diplomats want to sit at a tables with three seats such that by the end of the conference, each diplomat spoke a different language each day at lunch. Is this possible?

For some cases, the Baranyai theorem implies the Erdős-Ko-Rado theorem. This leads us to ask whether there is a way to expand the Baranyai theorem so that it implies the Erdős-Ko-Rado theorem in general.

Ideally, all material discussed in this thesis would be accessible to a careful reader who has taken a semester or more of proof based mathematics. In some places, a background in linear algebra is also necessary. We will begin by discussing some applications of hypergraphs to other sciences.

Many problems in extremal combinatorics originate from other areas of study because scientists often model systems in the real world using graphs. A *graph* is a collection of *vertices* (which can be thought of as points or nodes) and *edges* which join any two of the vertices. In recent years, some scientists have started using hypergraphs to describe these systems as well, because hypergraphs can convey more information than a graph can. A *hypergraph* is like a graph, except that a single edge can connect any number of vertices; we think of the vertices as being contained in the edge. We will illustrate examples in which scientists have used hypergraphs to successfully study biological networks, social networks, and computer science.

In [25], Klamt, Haus, and Theis introduce hypergraphs as a way to study biological networks that have previously been studied with graphs, but are limited by these models. For example, biologists model ecosystems as a set of species with interactions, and they model proteins as a networks of amino acids. Other examples include neural networks and food webs. Usually, these networks are represented by graphs, where the vertices represent units in the network, and edges represent interactions among the units. However, using graphs to represent these complex interactions can have limitations. Many relationships in biological networks are more complicated than what can be represented by a graph. If some process has more than two participants, then this relationship cannot be represented by a graph. Hypergraphs, however, can resolve this issue. For example, in protein-protein interaction networks, we have a set of proteins and a set of complexes (a group of associated proteins). Here, the proteins would form the vertices of a hypergraph, and the complexes would form the edges of the hypergraph. A problem related to experimental design is to determine the minimal subset of proteins that would cover all complexes. Formally, if X is the set of proteins and $\mathcal{H}$ is the collection of complexes, we are looking for a set $M \subset X$ such that for any $h \in \mathcal{H}$, there is an $m \in M$ such that $m \in h$. This problem can not be solved with a graph, but can be solved with a hypergraph.

In [11], Estrada and Rodríguez-Velázquez study complex networks with hypergraphs. Complex networks appear in almost all sciences, and usually they are represented by directed graphs, where the vertices are people, molecules, or computers, and the edges indicate some relationship between them. This includes the Internet, social networks, food webs, metabolic networks, and protein-protein interaction networks. Frequently, graphs do not provide a complete description of the relationships. For example, let a *collaboration network* consist of a collection of authors, the vertices, and let two authors share an edge whenever they have co-authored a paper. For example, in Figure 1, we have the collaboration network between Erdős, Bollobás, Daykin, Frankl, Lovász, and Katona. However, from this graph we only know whether a pair of mathematicians has collaborated. We cannot tell whether Bollobás and Daykin have published a paper together without Erdős, since the edge between Bollobás and Daykin may result from a paper published by Bollobás, Daykin and Erdős together. We also do not know if Erdős, Frankl, and Lovász have published a paper together. A natural way to fix this problem would be to represent such systems with hypergraphs, as we have done in Figure 2. This tells us that Bollobás, Daykin and Erdős had a paper together, but Bollobás and Daykin have not written a paper with just the two of them. On the other hand, Erdős, Frankl, and Lovász do not have a paper together, but each pair of them does have a paper together.

Estrada and Rodríguez-Velázquez studied one such collaboration network, and created the corresponding hypernetwork. When determining the most "central" author for each network, they came up with completely different results for the two models because an author who participates in many different collaboration groups may not have many coauthors if the groups are small, while an author with many coauthors may not have many collaboration groups if all coauthors are in one group.

The authors also studied food webs in ecosystems. A *competition graph* has as

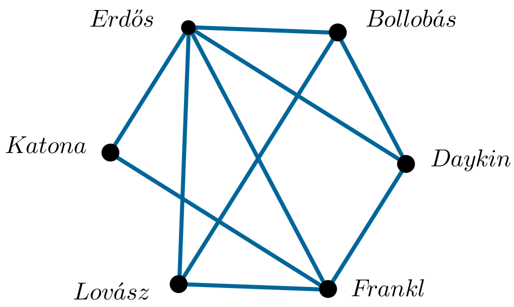


Figure 1: A collaboration network.

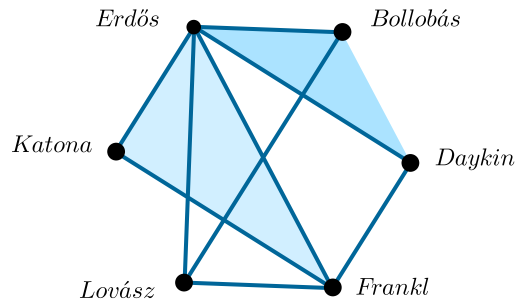


Figure 2: A collaboration hypernetwork.

its vertices the species in an ecosystem, and two vertices share an edge when they compete for the same prey. The problem with this model is that it does not give any information about when a group of species competes for the same prey, while a hypergraph can represent this type of information.

In [28], Liu and Wu apply hypergraphs to the declustering problem. The declustering problem is to partition data across multiple disks which can be accessed in parallel to reduce query response time. There are many different types of systems that need to access large amounts of complex data. Liu and Wu propose a hypergraph model to formulate the declustering problem in very general cases, where data items may have different sizes or queries might have different access frequencies.

To do this, they define a weighted hypergraph, where the vertex set is the set of data items, and the edges of the hypergraph each correspond to one of the queries, and consist of the set of data items that must be accessed in parallel for that query. Each edge is then weighted with the frequency of the query. Then the problem is to find a partition of the vertices which respects the disk capacities and minimizes the expected query response time. The authors show that hypergraph declustering is an NP complete problem, therefore any fast algorithm will not give the optimal solution. They use a greedy algorithm to find a heuristic solution.

The authors conduct experiments to compare the performance of their method against various other declustering methods. They vary the number of disks, the number of data items, and the number of queries. In almost all cases, they find that the hypergraph models outperform the other methods.

Outline

Chapter 1 contains the necessary definitions, and gives a brief discussion of the Erdős-Ko-Rado theorem and the Baranyai theorem.

Chapter 2 gives all seven known proofs of the Erdős-Ko-Rado theorem, each using different ideas, including permutations, linear algebra, linearly independent polynomials, and the Kruskal-Katona theorem.

Chapter 3 discusses several extensions and applications of the Erdős-Ko-Rado theorem. This includes a study of extremal sets under the Erdős-Ko-Rado conditions, a rephrasing of the Erdős-Ko-Rado theorem in terms of independent sets and an exploration of higher orders of independence, a discussion of Sperner sets, and a lower bound on the chromatic number of the Kneser graph.

Chapter 4 gives the complete proof of the Baranyai theorem and its corollaries, including Baranyai's "integer making" lemmas and his induction step.

Chapter 5 presents the wreath conjecture and discusses recent developments in that area, including a connected version of Baranyai's theorem, and a decomposition of complete uniform hypergraphs into Hamilton-Berge cycles. We briefly discuss the possibility that wreath decomposition is NP-complete, and discuss a reformulation of the problem in terms of graphs.

Chapter 6 explains why we do not think that the Erdős-Ko-Rado theorem can be proved from the Baranyai theorem, and also shows that the wreath conjecture implies the Erdős-Ko-Rado theorem.

Appendix A contains the *Mathematica* code used for the computations in this thesis.

Notation

Symbol	Meaning
$\|A\|$	The cardinality of a set A.
$\mathcal{P}(A)$	The power set of A, or the collection of all subsets of the set A.
K_n^k	The set of all k-subsets of the set $\{1, \ldots, n\}$.
$[n]$	The set $\{1, \ldots, n\}$.
(a, b)	The greatest common divisor of a and b.
$\binom{n}{k}$	The binomial coefficient, "n choose k".
$v_{\mathcal{H}}(x)$	The degree of the vertex x in the hypergraph $\mathcal{H}$.
Sij	The vertex exchange operation (see Chapter 2 Section 1).
$\delta_l(\mathcal{H})$	The l-shadow of the hypergraph $\mathcal{H}$ (see Chapter 2 Section 3).
X	Usually denotes the ground set or vertex set.
n	Usually denotes the size of the vertex set.
k, l, r, s, d	Usually denote integers.
i, j	Usually denote indices.
$\mathcal{H}, \mathcal{F}$	Usually denote hypergraphs.
e, f, g	Usually denote edges of a hypergraph.
x, v, w	Usually denote vertices of a hypergraph.

Chapter 1

Definitions and Setup

We begin with a formal discussion of the basic definitions, providing examples where appropriate. Then, we will introduce the Erdős-Ko-Rado theorem and the Baranyai theorem.

1.1 Hypergraphs

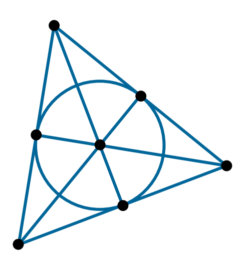
Let k be a positive integer. If X is a set, and $|X| = n$, then we will denote the set of all k-subsets (subsets of size k) by K_n^k. Let $\mathcal{P}(X)$ denote the power set of X.

Definition 1.1.1. A *hypergraph on* X is a set $\mathcal{H} \subset \mathcal{P}(X)$. A k*-uniform hypergraph* is a hypergraph $\mathcal{H}$ such that $\mathcal{H} \subset K_n^k$.

In this new framework, a usual combinatorial graph is a 2-uniform hypergraph. We will call X the *ground set* or *vertex set* of $\mathcal{H}$, and elements of X will be called *vertices*. We will call $\mathcal{H}$ the *edge set*, and elements of $\mathcal{H}$ will be called *hyperedges* or simply *edges*. We will usually use the letters e, f, g to denote edges, and x, v for vertices.

Definition 1.1.2. Let $\mathcal{H}$ be a hypergraph on X and let $x \in X$. The *degree* or *valency* of x is $v_{\mathcal{H}}(x) = |\{e \in \mathcal{H} \mid x \in e\}|$, or simply $v(x)$ when the context is clear. We will call $\mathcal{H}$ a d*-regular* hypergraph if $v_{\mathcal{H}}(x) = v_{\mathcal{H}}(x') = d$ for all $x, x' \in X$. We will call $\mathcal{H}$ an *almost regular* hypergraph if the valencies of any two vertices differ by at most 1.

Example The *Fano plane* is the hypergraph displayed below, where each of the six straight lines represent a single edge, and the circle represents an edge. The Fano plane is a 3-uniform, 3-regular intersecting hypergraph on seven vertices with seven edges.



1.2 Intersecting Hypergraphs

Recall the example from the introduction about diplomats: suppose that the world has n languages, and you wish to assemble a group of diplomats such that each diplomat in the group speaks exactly k languages, and any two diplomats speak a common language. Furthermore, no two diplomats speak exactly the same set of languages. What is the maximal size of the group of diplomats? This question can be answered by the Erdős-Ko-Rado Theorem, which is a central theorem from extremal combinatorics concerning the maximum number of edges in an intersecting hypergraph.

Definition 1.2.1. A hypergraph $\mathcal{H}$ is called *intersecting* if for all $e_1, e_2 \in \mathcal{H}$, we have that $|e_1 \cap e_2| \geq 1$. We say that $\mathcal{H}$ is *l-intersecting* if for all $e_1, e_2 \in \mathcal{H}$, we have that $|e_1 \cap e_2| \geq l$.

Relating this back to the example about diplomats, we would consider a hypergraph $\mathcal{H}$ on X where X is the set of languages, and each edge in $\mathcal{H}$ is the set of languages known by a diplomat. Then, the question of how many diplomats we may have translates to a question about how large an intersecting hypergraph can be. In the case when $\mathcal{H}$ is not necessarily uniform, or the diplomats can each know an arbitrary number of languages, we claim that $|\mathcal{H}| \leq 2^{n-1}$, where $n = |X|$. To see this, place the elements of $\mathcal{P}(X)$ into pairs $\{e, X \backslash e\}$ for each $e \in \mathcal{P}(X)$. Any intersecting hypergraph may have at most one element of each of these pairs present in its edge set. Since $|\mathcal{P}(X)| = 2^n$, there are 2^{n-1} pairs and so $|\mathcal{H}| \leq 2^{n-1}$. Note that we may achieve this upper bound by taking some $v \in X$, and letting the edge set be all subsets of X which contain v. It turns out that this is the maximal configuration, so in the largest possible collection of diplomats, all diplomats speak a common language.

Now consider the case when $\mathcal{H}$ is a k-uniform hypergraph, or each diplomat speaks exactly k languages. If $n < 2k$, then any subset of $X^{(k)}$ will be intersecting, so we can have $\mathcal{H} = K_n^k$. If $n = 2k$, then we may again pair the edges in K_n^k into $\{e, X \backslash e\}$ pairs. Then an intersecting hypergraph may have at most one edge from each pair, for an upper bound of

$$\frac{1}{2}\binom{n}{k} = \binom{n-1}{k-1}$$

edges.

What happens when $n > 2k$? Consider a hypergraph $\mathcal{H}$ where for some $v \in X$, we have that

$$\mathcal{H} = \{e \in K_n^k \mid v \in e\}.$$

Then $\mathcal{H}$ is a k-uniform intersecting hypergraph with $\binom{n-1}{k-1}$ edges. Is this maximal, and are there any other constructions which yield a hypergraph of this size?

Theorem 1.2.2 (Erdős, Ko, Rado [10])**.** *Let $k \geq 2$, $n > 2k$, and $\mathcal{H}$ be an intersecting k-uniform hypergraph on X, with $n = |X|$. Then*

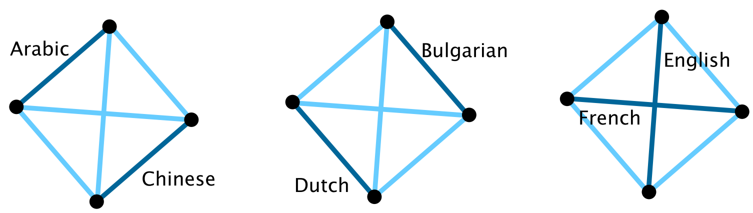
$$|\mathcal{H}| \leq \binom{n-1}{k-1},$$

with equality if and only if for some $v \in X$, we have $\mathcal{H} = \{e \in K_n^k \mid v \in e\}$.

There are many proofs of this theorem, which we will discuss in Chapter 2. For now, we will introduce the Baranyai theorem.

1.3 Factorizations of Hypergraphs

We now reconsider the diplomat seating problem given in the introduction: suppose that we have a group of n diplomats such that any set of k diplomats shares a language, and each language is spoken by exactly k diplomats. Then there are $\binom{n}{k}$ languages spoken in total. The diplomats are attending a conference which lasts $\binom{n-1}{k-1}$ days. During lunch, each diplomat is asked to sit at a table with k seats such that by the end of the conference, each diplomat spoke a different language each day at lunch. Is this possible? Clearly, we must require that k divides n, because the n diplomats must split evenly into groups of size k each day for lunch. When this condition is satisfied, there will be n/k tables at lunch each day, and since $(n/k)\binom{n-1}{k-1} = \binom{n}{k}$, each language will be spoken at exactly one table on one day of the conference. From the outset, it seems possible to find a seating arrangement for the diplomats. For example, if there are 4 diplomats and each pair of diplomats speaks a different language, then we have the scenario in Figure 1.1.



Day 1 *Day* 2 *Day* 3

Figure 1.1: A solution for the diplomat seating problem when $n = 4$ and $k = 2$.

It was proved by Baranyai in 1975 that as long as k divides n, this problem has a solution. It turns out that to solve the problem, we must find a *factorization* of K_n^k.

Definition 1.3.1. Let d be a positive integer. A *d-factor* of a hypergraph $\mathcal{H}$ is a set of edges $\mathcal{H}' \subset \mathcal{H}$ such that $v_{\mathcal{H}'}(x) = d$ for all $x \in X$. A *d-factorization* of $\mathcal{H}$ is a partition of $\mathcal{H}$ into d-factors:

$$\mathcal{H} = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_l,$$

where the $\mathcal{H}_i$ are d-factors and $\mathcal{H}_i \cap \mathcal{H}_j = \emptyset$ for $i \neq j$. If $\mathcal{H}$ is 1-factorizable, we will simply call $\mathcal{H}$ factorizable, and its 1-factorization will be called a factorization.

Returning to our example, Figure 1.1 gives a factorization of K_4^2, and the seating arrangement on a given day is a factor. Furthermore, the graph K_{2n}^2 can be factorized by arranging $2n-1$ of the vertices into a regular $(2n-1)$-gon, and leaving the excluded vertex to the outside. Then, for each factor, we draw parallel lines connecting vertices in the polygon. This will leave one leftover vertex from inside the polygon, which will be connected to the excluded vertex. Doing this in all possible ways gives a factorization. See Figure 1.2 for such a factorization of K_6^2.

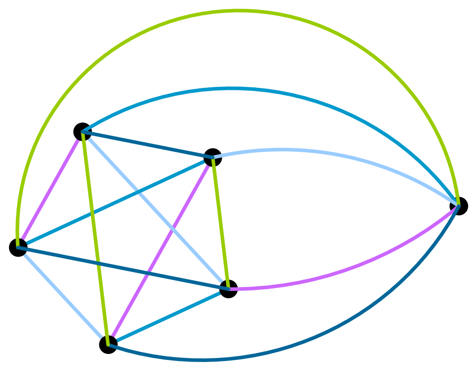


Figure 1.2: A 1-factorization of K_6^2, where each color represents a factor.

The graph K_{2n+1}^2 can be 2-factored by making the factors cycles in the graph. See Figure 1.3 for a 2-factorization of K_7^2.

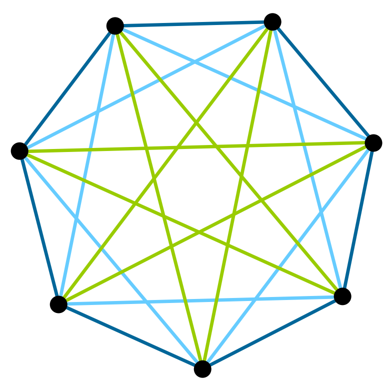


Figure 1.3: A 2-factorization of K_7^2, where each color represents a factor.

Note that if K_n^k is factorizable then k divides n, as we observed before. It turns out that this necessary condition is also sufficient. This assertion had been around in a vague form for more than 100 years before it was proved by Baranyai.

Theorem 1.3.2 (Baranyai [3]). *The complete graph K_n^k is factorizable if and only if k divides n.*

The proof of the Baranyai theorem is quite difficult and will be the subject of Chapter 4.

Now we will show that the Baranyai theorem immediately implies the Erdős-Ko-Rado theorem in the case where k divides n. Consider $\mathcal{H}$, a k-uniform hypergraph on n vertices. Let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_l$ be a factorization of K_n^k. Then $\mathcal{H}$ contatins at most one edge from each of the factors $\mathcal{H}_i$, so $|\mathcal{H}| \leq l$, and

$$l = \binom{n}{k}\frac{k}{n} = \binom{n-1}{k-1},$$

so we have that $|\mathcal{H}| \leq \binom{n-1}{k-1}$.

This leads us to ask the following question: is there a way to expand this proof to show that the Baranyai theorem implies the Erdős-Ko-Rado theorem in general? Is there a generalization of the Baranyai theorem that implies the Erdős-Ko-Rado theorem in all cases? These are the questions we wish to answer in this thesis.

Chapter 2

Proofs of the Erdős-Ko-Rado Theorem

Erdős had many publications with Rado, and began corresponding with him in early 1934 when Rado was a German refugee in Cambridge and Erdős was in Budapest [9]. Their first joint paper was done with Chao Ko, and contained the Erdős-Ko-Rado Theorem. Erdős writes that the paper was essentially finished in 1938, and that one of the reasons it was not published until 1961 was that at the time there was relatively little interest in combinatorics. Another reason was that in 1938, the three separated: Ko returned to China, Erdős went to Princeton, and Rado stayed in England. Erdős says that the Erdős-Ko-Rado Theorem is "perhaps our most quoted result"[9]. The Erdős-Ko-Rado Theorem, referenced by hundreds of other papers, opened the way for the rapid development of extremal combinatorics.

Recall from the introduction that the Erdős-Ko-Rado Theorem states that if $k \geq 2$, $n > 2k$, and $\mathcal{H}$ is an intersecting k-uniform hypergraph on n vertices, then

$$|\mathcal{H}| \leq \binom{n-1}{k-1}.$$

There are many known proofs of this theorem, some of which are outlined in [16]. Here we will present all currently known proofs.

2.1 Shifting: The Original Proof

The original proof of the theorem given by Erdős, Ko, and Rado in [10] uses an idea called *shifting*, which replaces the edges of a hypergraph with new ones such that some key properties are preserved.

Definition 2.1.1. Let $\mathcal{H} \subset \mathcal{P}(X)$, where $X = \{x_1, \ldots, x_n\}$. Let $1 \leq i < j \leq n$. The *exchange operation* $S_{ij} : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by

$$S_{ij}(f) = \begin{cases} \{x_i\} \cup (f - \{x_j\}) & \text{if } x_i \notin f,\, x_j \in f, \text{ and } \{x_i\} \cup (f - \{x_j\}) \notin \mathcal{H} \\ f & \text{otherwise.} \end{cases}$$

Then we define $S_{ij}(\mathcal{H}) = \{S_{ij}(f) \mid f \in \mathcal{H}\}$.

We note that $|S_{ij}(\mathcal{H})| = |\mathcal{H}|$, and that $|S_{ij}(f)| = |f|$ for all $f \in \mathcal{H}$. See Appendix A.3 for *Mathematica* code which performs the exchange operation.

Proposition 2.1.2. *If $\mathcal{H} \subset \mathcal{P}(X)$, and $\mathcal{H}$ is l-intersecting, then so is $S_{ij}(\mathcal{H})$.*

Proof. Let $f_1, f_2 \in S_{ij}(\mathcal{H})$. Then there are several possibilities.

1. If $f_1, f_2 \in \mathcal{H}$, then $|f_1 \cap f_2| \geq l$.

2. If $f_1 \in \mathcal{H}$, and $f_2 \notin \mathcal{H}$, then $f_2 = \{x_i\} \cup (f_2' \backslash \{x_j\})$ for some $f_2' \in \mathcal{H}$. Regarding f_1, one of the following is true. If $x_i \in f_1$, then $|f_2 \cap f_1| \geq 1 + |f_2' \cap f_1| - 1 = l$. If $x_j \notin f_1$, then $|f_1 \cap f_2| \geq |f_2' \cap f_1| = l$. Lastly, if $\{x_i\} \cup (f_1 \backslash \{x_j\}) \in \mathcal{H}$, then since $|f_2' \cap (\{x_i\} \cup (f_1 \backslash \{x_j\}))| \geq l$, we have that $l + 1 \leq |f_2' \cap f_1| = |f_2 \cap f_1| + 1$.

3. Lastly, if $f_1, f_2 \notin \mathcal{H}$, then $|f_1 \cap f_2| = |f_1' \cap f_2'| \geq l$.

Therefore, $S_{ij}(\mathcal{H})$ is l-intersecting. □

Proof of the theorem. Let $k \geq 2$, $n > 2k$, and $\mathcal{H}$ be an intersecting k-uniform hypergraph on X, with $n = |X|$. The original proof applied induction on n, and proved the theorem for all $k \leq n/2$. The base case, $n = 2k$, was discussed in Chapter 1. Now, suppose that $n > 2k$, and that the theorem is true for all smaller n. Define inductively $\mathcal{H}_0 = \mathcal{H}$, and $\mathcal{H}_i = S_{in}(\mathcal{H}_{i-1})$, for any $i \in \{0, \ldots, n-1\}$. Then by the proposition, $|\mathcal{H}| = |\mathcal{H}_{n-1}|$, and $\mathcal{H}_{n-1} \subset K_n^k$ is intersecting.

Define $G = \{f \in \mathcal{H}_{n-1} \mid x_n \notin f\}$, and $F = \{f \backslash \{x_n\} \mid x_n \in f \in \mathcal{H}_{n-1}\}$. Clearly, $|\mathcal{H}| = |G| + |F|$. Then since G is intersecting, and has $n-1$ vertices, we have by the inductive hypothesis that

$$|G| \leq \binom{n-2}{k-1}.$$

Then it suffices to show that $|F| \leq \binom{n-1}{k-2}$, since this would imply $|\mathcal{H}| \leq \binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1}$.

Since F is a $(k-1)$-uniform hypergraph on $n-1$ vertices, if F is intersecting then the theorem is proved by the inductive hypothesis. Suppose that there are $f, f' \in F$ such that $f \cap f' = \emptyset$. Since $|f \cup f'| = 2(k-1) < n-1$, there is an x_i, where $i < n$, such that $x_i \notin f \cup f'$. Let $h = f \cup \{x_n\}$. Then $h \in \mathcal{H}_{n-1}$. Since $x_n \in h$, we have that $h \in \mathcal{H}$ (because x_n was never shifted out). Therefore, $h \in \mathcal{H}_i$ for all $1 \leq i \leq n-1$. Then $S_{in}(h) = h$, so h was never replaced. This can happen only if $(f \cup \{x_i\}) \in \mathcal{H}_{i-1}$ (since $x_i \notin h, x_n \in h$). However, $(f \cup \{x_i\}) \cap (f' \cup \{x_n\}) = \emptyset$, which is a contradiction because they are both members of the intersecting hypergraph $\mathcal{H}_{n-1}$. □

2.2 Cyclic Permutations: Katona's Proof

This clever proof was given by Katona in [23]. It works by counting in two ways the number of *cyclic extensions* of an edge.

Proof. Let $\mathcal{H}$ be intersecting, k-uniform, on n vertices, and let $n \geq 2k+1$. We will say that a cyclic permutation of the vertices of $\mathcal{H}$ *extends an edge* $e \in \mathcal{H}$ if the vertices of e appear in consecutive positions of the permutation when the permutation is written in cycle notation. For example, the edge $\{x_1, x_2, x_3\}$ is extended by the permutation $(x_4\ x_5\ x_1\ x_3\ x_2)$, but is not extended by the permutation $(x_4\ x_1\ x_5\ x_3\ x_2)$.

There are $k!(n-k)!$ cyclic permutations extending an edge e. On the other hand, a cyclic permutation can be the extension of at most k edges. To see this, suppose $(x_1, \ldots, x_k)$ is one of the edges in the cycle. Then for all i, $1 \leq i \leq k-1$, there is at most 1 edge which contains precisely one of $\{x_i, x_{i+1}\}$ by the intersecting property.

The number of cyclic permutations of $\{1, \ldots, n\}$ is $(n-1)!$, so

$$\underbrace{|\mathcal{H}|k!(n-k)!}_{\#\text{ cyclic extensions}} \quad \leq \quad \underbrace{k(n-1)!}_{\#\text{ potential cyclic extensions}}$$

Then,

$$\begin{aligned}|\mathcal{H}| &\leq \frac{k(n-1)!}{k!(n-k)!} \\ &= \frac{(n-1)!}{(k-1)!(n-k)!} \\ &= \binom{n-1}{k-1}.\end{aligned}$$

□

2.3 Shadows: Daykin's Proof

Daykin uses a version of the Kruskal-Katona theorem to prove the Erdős-Ko-Rado theorem. First, we will define a shadow.

Definition 2.3.1. Given a k-uniform hypergraph $\mathcal{H}$ and an integer l, where $1 \leq l \leq k$, the *l-shadow* δ_l of $\mathcal{H}$ will be:

$$\delta_l(\mathcal{H}) = \{g \in K_n^l \mid \text{for some } f \in \mathcal{H}, \text{ we have } g \subset f\}.$$

Given $|\mathcal{H}| = m$, what can we say about the size of $\delta_l(\mathcal{H})$? It is clear that $|\delta_l(\mathcal{H})| \leq \binom{k}{l}|\mathcal{H}|$, where equality holds if and only if $|f \cap f'| < l$ holds for all distinct $f, f' \in \mathcal{H}$. Can we get a lower bound for $\delta_l(\mathcal{H})$? The following consequence of the Kruskal-Katona theorem by Lovász answers this question.

Theorem 2.3.2 (Kruskal- Katona [26, 22])**.** *Let $\mathcal{H}$ be a k-uniform hypergraph, and suppose $|\mathcal{H}| \geq \binom{x}{k}$ for some $x \geq k$. Then*

$$|\delta_l(\mathcal{H})| \geq \binom{x}{l} \qquad \textit{for all } 0 \leq l \leq k.$$

We will present a proof of Theorem 2.3.2 which was given by Daykin in [6].

Proposition 2.3.3. *Let $\mathcal{H} \subset K_n^k$, and let $1 \leq i < j \leq n$. Then*

$$\delta_{k-1}(S_{ij}(\mathcal{H})) \subset S_{ij}(\delta_{k-1}(\mathcal{H}))$$

Proof. There are several cases which we must examine. Let $f \in \delta_{k-1}(S_{ij}(\mathcal{H}))$.

1. If $x_i, x_j \notin f$, then then $f \in \delta_{k-1}(\mathcal{H})$ (Since either $f \cup \{x_i\}$ or $f \cup \{x_j\}$ is in $\mathcal{H}$). Then since $x_j \notin f$, we have that $S_{ij}(f) = f$, so $f \in S_{ij}(\delta_{k-1}(\mathcal{H}))$.

2. If $x_j \in f$, and $x_i \in f$, then there exists a $v \in X$ such that $f \cup \{v\} \in S_{ij}(\mathcal{H})$. Since $x_i, x_j \in f$, we have that $f \cup \{v\} \in \mathcal{H}$. Then $f \in \delta_{k-1}(\mathcal{H})$, and since $x_i, x_j \in f$, we have that $f \in S_{ij}(\delta_{k-1}(\mathcal{H}))$.

3. If $x_j \in f$, and $x_i \notin f$, then there exists a $v \in X$ such that $f \cup \{v\} \in S_{ij}(\mathcal{H})$. Since $x_j \in f$, we have that $S_{ij}(f \cup \{v\}) = f \cup \{v\}$, so $f \cup \{v\} \in \mathcal{H}$. Therefore, $f \in \delta_{k-1}(\mathcal{H})$. Now, it matters what v from before was. If $v = x_i$, then $(f \backslash \{x_j\}) \cup \{x_i\} \in \delta_{k-1}(\mathcal{H})$, so $S_{ij}(f) = f$. If $v \neq x_i$, then there exists $f' \in \mathcal{H}$ such that $f' = ((f \cup \{v\}) \backslash \{x_j\}) \cup \{x_i\}$, so $f' \backslash \{v\} = (f \backslash \{x_j\}) \cup \{i\} \in \delta_{k-1}(\mathcal{H})$. Therefore, $S_{ij}(f) = f$, so $f \in S_{ij}(\delta_{k-1}(\mathcal{H}))$.

4. If $x_i \in f$, and $x_j \notin f$, then there exists a $v \in X$ such that $f \cup \{v\} \in S_{ij}(\mathcal{H})$. Either $f \cup \{v\} \in \mathcal{H}$ or $(f \backslash \{x_i\}) \cup \{v\} \cup \{x_j\} \in \mathcal{H}$. In the first case, $f \in \delta_{k-1}(\mathcal{H})$, and so since $x_j \notin f$, we have that $S_{ij}(f) = f$. In the other case, we have that $(f \backslash \{x_i\}) \cup \{x_j\} \in \delta_{k-1}(\mathcal{H})$. Therefore, in either case, $f \in S_{ij}(\delta_{k-1}(\mathcal{H}))$.

□

Continuing with the proof of the Kruskal-Katona theorem, we define inductively $\mathcal{H}_1 = \mathcal{H}$, and $\mathcal{H}_i = S_{1i}(\mathcal{H}_{i-1})$, for $2 \leq i \leq n$. Then, we then have that

$$|\delta_{k-1}(\mathcal{H}_n)| \leq |\delta_{k-1}(\mathcal{H})|.$$

Therefore, we may consider $\mathcal{H}_n$ instead.

Claim 2.3.4.

1. $|\delta_{k-1}(\mathcal{H}_n)| \geq |\mathcal{H}_n(1)| + |\delta_{k-1}(\mathcal{H}_n(1))|$, *where* $\mathcal{H}_n(1) = \{e \backslash \{x_1\} \in \mathcal{H}_n \mid x_1 \in e\}$.

2. $\delta_{k-1}(\mathcal{H}_n(\bar{1})) \subset \mathcal{H}_n(1)$, *where* $\mathcal{H}_n(\bar{1}) = \{e \in \mathcal{H}_n \mid x_1 \notin \mathcal{H}\}$.

Proof.

1. By definition, $\mathcal{H}_n(1) \subset \delta_{k-1}(\mathcal{H}_n)$, and $\{\{x_1\} \cup g \mid g \in \delta_{k-1}(\mathcal{H}_n(1))\} \subset \delta_{k-1}(\mathcal{H}_n)$. These families are disjoint, so the claim holds.

2. Choose (g, h) with $h \in \mathcal{H}_n(\bar{1})$ and $g \subset h$ such that $|g| = k - 1$. Let $\{x_i\} = h \backslash g$. The only way h survived S_{1i} is if $h' = g \cup \{x_1\} \in \mathcal{H}_{i-1}$, so that $h' \in \mathcal{H}_n$. Therefore, $g \in \mathcal{H}_n(1)$.

□

Proof of Kruskal-Katona. Apply induction on k and for a given k, apply induction on $|\mathcal{H}|$ as well. The case where $|\mathcal{H}| = 1$ is trivial. First, suppose that $|\mathcal{H}_n(1)| < \binom{x-1}{k-1}$. By 2 from Claim 1 above, $|\mathcal{H}_n(1)| \geq k$, so we ensure that $x - 1 > k$. Now, notice that

$$|\mathcal{H}_n(\bar{1})| = |\mathcal{H}| - |\mathcal{H}_n(1)| \geq \binom{x}{k} - \binom{x-1}{k-1} = \binom{x-1}{k}.$$

Applying the induction hypothesis and using 2 again, we find that

$$|\mathcal{H}_n(1)| \geq |\delta_{k-1}(\mathcal{H}_n(\bar{1}))| \geq \binom{x-1}{k-1},$$

which is a contradiction.

Therefore, $|\mathcal{H}_n(1)| \geq \binom{x-1}{k-1}$. By inductive the hypothesis, we know that $|\delta_{k-2}(\mathcal{H}_n(1))| \geq \binom{x-1}{k-2}$, and so by (1) in the claim above, we have

$$\begin{aligned}
|\delta_{k-1}(\mathcal{H}_n)| &\geq |\mathcal{H}_n(1)| + |\delta_{k-2}(\mathcal{H}_n(1))| \\
&= \binom{x-1}{k-1} + \binom{x-1}{k-2} \\
&= \binom{x}{k-1}.
\end{aligned}$$

□

Now, we will prove the Erdős-Ko-Rado theorem from the Kruskal-Katona theorem. This proof was given in [5].

Proof of Erdős-Ko-Rado. Suppose that $\mathcal{H} \subset K_n^k$, with $n \geq 2k$, and $\mathcal{H} > \binom{n-1}{k-1} = \binom{n-1}{n-k}$. Define $G = \{X \backslash f \mid f \in \mathcal{H}\} \subset K_n^{n-k}$. By the Kruskal-Katona theorem, we have that $|\delta_k(G)| \geq \binom{n-1}{k}$, so $|\mathcal{H}| + |\delta_k(G)| > \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$. Therefore, there is an $f \in \mathcal{H} \cap \delta_k(G)$. Since $f \in \delta_k(G)$, we have that $f \subset X \backslash f'$ for some $f' \in \mathcal{H}$, so $f \cap f' = \emptyset$. But this is a contradiction, since $\mathcal{H}$ is intersecting. □

2.4 Linear Algebra: Lovász' Proof

This proof by Lovász appears in [30], and utilizes linear algebra to prove the Erdős-Ko-Rado theorem.

Let $N = \binom{n}{k}$, and $A_1, \ldots, A_N$ be an arbitrary ordering of the k-subsets of X. For an intersecting hypergraph $\mathcal{H} \subset K_n^k$, let

$$\chi(\mathcal{H}) = (\chi_1, \ldots, \chi_N)$$

be its characteristic vector: $\chi_i = 1$ if $A_i \in \mathcal{H}$, and $\chi_i = 0$ if $A_i \notin \mathcal{H}$. Let B be any $N \times N$ real symmetric matrix whose entries b_{ij} are 0 whenever $A_i \cap A_j \neq \emptyset$. Let I be the $N \times N$ identity matrix, and let J be the $N \times N$ matrix whose entries are all 1s.

Claim 2.4.1. *If $B + I - cJ$ is positive semidefinite for some $c > 0$, then $|\mathcal{H}| \leq 1/c$.*

Proof. Let $w = \chi(\mathcal{H})$, and consider $y = w(B+I-cJ)w^T$. By assumption, $wBw^T = 0$. Also, $wIw^T = |\mathcal{H}|$, and $wJw^T = |\mathcal{H}|^2$. Therefore, $y = |\mathcal{H}| - c|\mathcal{H}|^2$. By semi-definiteness, we have that $y \geq 0$, so that $c|\mathcal{H}|^2 \leq |\mathcal{H}|$, or $|\mathcal{H}| \leq 1/c$. $\square$

Now, to prove the theorem, we only need to show that for some choice of the matrix B, and for $c = \binom{n-1}{k-1}^{-1}$, we have that $B+I-cJ$ is positive semidefinite. Then define the matrix $B = (b_{ij})$ by

$$b_{ij} = \begin{cases} \binom{n-k-1}{k-1}^{-1} & A_i \cap A_j = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

We will compute the eigenvalues of B.

First, the vector $u = (1, \ldots, 1)$ is a common eigenvector of B, I, and J. The eigenvalue of B is $\frac{n-k}{k}$, since A_i is disjoint from $\binom{n-k}{k}$ edges, we get $\binom{n-k}{k}\binom{n-k-1}{k-1}^{-1} = \frac{n-k}{k}$. The eigenvalue for I is clearly 1, and for J it is $\binom{n}{k}$. The all 1's vector u is annihilated by $B + I - cJ$:

$$Bu + Iu - cJu = \frac{n-k}{k}u + u - \binom{n-1}{k-1}^{-1}\binom{n}{k}u = \frac{n}{k}u - \frac{n}{k}u = 0.$$

Therefore, the eigenvalue of u for $B + I - cJ$ is 0.

Now, we will find the remaining eigenvectors v of B, and show that the following hold:

1. The eigenvalue for I of v is 1 (this is true for any v).

2. An eigenvector for J is v, with eigenvalue 0.

3. An eigenvector for B is v, with eigenvalue greater than or equal to -1.

Together, these statements ensure that all eigenvalues of $B + I - cJ$ are greater than or equal to 0, which gives positive semi definiteness for $B + I - cJ$.

The following collection of vectors have eigenvalue -1 with B. For each (x, y), where $x, y \in X$, define $v(x, y) = (v_1, \ldots, v_N)$ with

$$v_i = \begin{cases} 1 & \text{if } A_i \cap \{x, y\} = \{x\} \\ -1 & \text{if } A_i \cap \{x, y\} = \{y\} \\ 0 & \text{otherwise.} \end{cases}$$

First, note that these are eigenvectors of J with eigenvalue 0.

Claim 2.4.2. *B $v(x,y)$ = -$v(x,y)$*

Proof. If B_i is the ith row of B, then the ith entry of $Bv(x, y)$ is $B_i \cdot v(x, y)$.

If $A_i \cap \{x, y\} = \{x, y\}$, then A_i is only disjoint from edges not containing $\{x, y\}$, so only those entries will be nonzero in B_i, but in $v(x, y)$, those entries will be 0. Therefore, $B_i \cdot v(x, y) = 0 = v_i$.

If $A_i \cap \{x, y\} = \emptyset$, then A_i is disjoint from an equal number of edges containing x and edges containing y, so in $B \cdot v(x, y)$ these terms cancel, so that $B \cdot v(x, y) = 0 = v_i$.

If $A_i \cap \{x, y\} = \{x\}$, then any edge disjoint from A_i does not contain x, so there is a nonzero term in $B \cdot v(x, y)$ if and only if $y \in A_j$. There are $\binom{n-k-1}{k-1}$ of these vectors, so that

$$B \cdot v(x, y) = -1 \cdot \binom{n-k-1}{k-1}\binom{n-k-1}{k-1}^{-1} = -1 = -v_i,$$

and similarly for the case where $A_i \cap \{x, y\} = \{y\}$.

Therefore, $v(x, y)$ is an eigenvector of B with eigenvalue -1. □

The $v(x, y)$ span a $n - 1$ dimensional vector space. To see this, let $z \in X$. Form a basis for the subspace spanned by $v(x, y)$ by taking the set $\beta = \{v(z, x) \mid x \in X\}$. Then, for any $y_1, y_2 \in X$, it is easy to verify that $v(y_1, y_2) = v(z, y_2) - v(z, y_1)$. Then β is clearly linearly independent, and a spanning set for the $v(x, y)$, and $|\beta| = n - 1$.

Then what are the remaining $\binom{n}{k} - (n-1) - 1$ eigenvectors for B? For $2 \le i \le k$, and any two disjoint i-element sets of X,

$$C = \{x_1, \ldots, x_i\} \qquad D = \{y_1, \ldots, y_i\},$$

Define $u(C, D) = \{u_1, \ldots, u_N\}$ by

$$u_j = \begin{cases} (-1)^{|D \cap A_j|} & \text{if } |A_j \cap \{x_l, y_l\}| = 1 \text{ holds for } 1 \le l \le i \\ 0 & \text{else.} \end{cases}$$

Claim 2.4.3. $Bu(C, D) = (-1)\frac{\binom{n-k-i}{k-i}}{\binom{n-k-1}{k-1}} u(C, D)$.

Proof. Set $\delta = \binom{n-k-1}{k-1}^{-1}$. Compute the rth entry v_r of $Bu(C, D)$. This is the dot product of u and the rth row of B.

Suppose first that $|A_r \cap \{x_l, y_l\}| = 1$ for $1 \le l \le i$. The only way to obtain a nonzero entry is for $A \in K_n^k$ to satisfy $A \cap (C \cup D) = (C \cup D) \backslash A_r$ (meaning A and A_r are complementary inside of $C \cup D$). Therefore,

$$v_r = (-1)\delta\binom{n-k-i}{k-i} u_r.$$

If $|A_r \cap \{x_l, y_l\}| = 2$ for some l, then there is no way to get a nonzero term, so $v_r = 0$.

Consider the case where $|A_r \cap \{x_l, y_l\}| = 0$. There are an equal number of A, A' such that $A \cap \{x_l, y_l\} = \{x_l\}$ and $A' \cap \{x_l, y_l\} = \{y_l\}$. Therefore, since these entries have opposite signs in $u(C, D)$, all terms cancel in the dot product, so $v_r = 0$. □

Now, we will find $\binom{n}{i} - \binom{n}{i-1}$ linearly independent vectors $u(C, D)$. This will show that the eigenvectors corresponding to the eigenvalue

$$(-1)^i \frac{\binom{n-k-i}{k-i}}{\delta}$$

span a vector space of dimension at least $\binom{n}{i} - \binom{n}{i-1}$. If $C = \{x_1, \ldots, x_i\}$ and $D = \{y_1, \ldots, y_i\}$ with $x_i < y_i$, then we will write $C < D$. Then there are exactly $\binom{n}{i} - \binom{n}{i-1}$ sets $D \in K_n^i$ for which some $C < D$ exists. For each D, find some $C = C(D)$ with this property. Then the vectors $u(C(D), D)$ are linearly independent.

Since these lower bounds on the dimensions of the eigenspaces sum to $\binom{n}{k}$, equality holds everywhere. Since $\binom{n-k-i}{k-i} \leq \binom{n-k-1}{k-1}$, we have that the eigenvectors of B all satisfy properties (1), (2), and (3) from above, so that $B + I - cJ$ is positive semidefinite, and the theorem holds.

2.5 Another proof by Katona

This proof is given by Katona in [21], and relies on the following theorem.

Theorem 2.5.1. *Let $1 \leq s \leq k$, and $1 \leq l \leq k$, and $s + l \geq k$, and let $\mathcal{H}$ be a hypergraph on n vertices with m edges such that $\mathcal{H}$ is k-uniform and l-intersecting. Then*

$$m \frac{\binom{2k-l}{s}}{\binom{2k-l}{k}} \leq |\delta_s(\mathcal{H})|$$

where $\delta_s(\mathcal{H})$ denotes the family of s-subsets of the edges of $\mathcal{H}$.

Proof. If $s = k$, then the theorem is clear, so we only consider the case when $s < k$. There will be 3 different cases.

1. $2k - l \geq n$.

   Let $c \in \delta_s\mathcal{H}$. Count the pairs (h_i, c), where $h_i \in H$, and $c \subset h_i$. On the one hand, there are $|\mathcal{H}|\binom{k}{s}$ of them, since there are $\binom{k}{s}$ possible choices of c for each choice of h_i, and there are $|\mathcal{H}|$ choices for h_i. On the other hand, there are $|\delta_s(\mathcal{H})|\binom{n-s}{k-s}$ of them, since for a fixed c there are possibly $\binom{n-s}{k-s}$ sets to join to c to obtain an h_i, and there are $|\delta_s(\mathcal{H})|$ choices for c. So, we have

   $$|\mathcal{H}|\binom{k}{s} \leq |\delta_s(\mathcal{H})|\binom{n-s}{k-s}.$$

   Then, we need to prove that

   $$\frac{\binom{k}{s}}{\binom{n-s}{k-s}} \geq \frac{\binom{2k-l}{s}}{\binom{2k-l}{k}}$$

   in order to prove the theorem. Simplifying this expression, we find that

   $$\frac{(2k-l-s)!}{(n-s)!} \geq \frac{(k-l)!}{(n-k)!}$$

Whenever $2k - l > n$, we see that the left hand side is greater than 1, and the right hand side is less than 1. Equality holds only in the case $2k - l = n$.

2. $s = 1$

   Here, there are 2 cases.

   (a) $l = k$

   If $l = k$, then $|\mathcal{H}| = 1$, so $n = k$. Then $2k - l = l = n$, and

   $$n = |\mathcal{H}| \frac{\binom{n}{1}}{\binom{n}{n}} \leq |\delta_1(\mathcal{H})| = n.$$

   (b) If $l = k - 1$ then we have 2 cases. Here, it is enough to show that $|\mathcal{H}| \leq |\delta_1(\mathcal{H})|$, since

   $$\frac{\binom{k+1}{1}}{\binom{k+1}{k}} = 1$$

   i. If every set of size $k - 1$ is in at most 2 edges then consider $h_i \cap h_j$ for $h_i, h_j \in \mathcal{H}$. Clearly,

   $$|(h_i \cap h_j) \cap h_a| < k - 1,$$

   however

   $$|(h_i \cap h_j) \cap h_a| < k - 2$$

   is not possible, since we have

   $$|h_i \cap h_a| \leq |(h_i \cap h_j) \cap h_a| + 1 < k - 1.$$

   So, for all $h_a \in \mathcal{H}$, we find that $|(h_i \cap h_j) \cap h_a| = k - 2$. Then we have $h_i \backslash h_j \subset h_a$ for all $h_a \in \mathcal{H}$, since $|h_i \cap h_j| = k - 1$ and $|(h_i \cap h_j) \cap h_a| = k - 2$. Similarly, we have $h_j \backslash h_i \subset h_a$ for all $h_a \in \mathcal{H}$. Then since the sets $h_i \cap h_j$, $h_i \backslash h_j$, and $h_j \backslash h_i$ are all disjoint, we have that

   $$h_a = (h_j \backslash h_i) \cup (h_i \backslash h_j) \cup (h_i \cap h_j) \backslash \lambda_a,$$

   where $\lambda_a \in h_i \cap h_j$. Then, $|\mathcal{H}| \leq k + 1$, and each element is in at most k sets h_a. Thus

   $$\mathcal{H} \leq |\delta_1(\mathcal{H})|.$$

   ii. If $\mathcal{H}$ is such that there exists a c, with $|c| = k - 1$, and $c \subset h_i, h_j, h_f$, then for any a, we have $c \subset h_a$. This is because $|c \cap h_a| < k - 2$ cannot hold, since in this case

   $$|h_i \cap h_a| < k - 1.$$

   Also, it cannot be the case that $|c \cap h_a| = k - 2$, since this would imply $h_a \supset h_i \backslash c$, and $h_a \supset h_j \backslash c$, and $h_a \supset h_f \backslash c$ because

   $$|h_i \cap h_j| = |h_j \cap h_f| = |h_f \cap h_i| = k - 1,$$

   so $|h_a| \geq k + 1$, which is impossible. Then we have $n = |\mathcal{H}| + k - 1$, so that $|\mathcal{H}| \leq |\delta_1(\mathcal{H})|$.

3. $2k - l < n$ and $s > 1$.

   We will induct over n, and apply the first two cases. Here, we have that $1 < s < k < n$, so $n \geq 3$. First we consider the case $n = 3$. Here, we must have $k = 3$, and $|\mathcal{H}| = 1$, and $s = 2, l = 1$ or 2. We do not worry about the case $l = 3$, since then $2k - l = n$, and this is handled already by case 1. Since $|\delta_2\mathcal{H}| = 3$ and $\binom{5}{2}/\binom{5}{3} = 1$, and $\binom{4}{2}/\binom{4}{3} = \frac{3}{2}$, in both cases strict equality holds.

   Suppose $n > 3$ and for $n - 1$, the theorem is true. Then we will prove the theorem for n. Let d_a be the sum of the indexes of h_a. We may assume that our system is such that $|\delta_s(\mathcal{H})|$ is minimal and amongst all such systems, $\sum_a d_a$ is minimal. We now distinguish two subclasses.

   (a) Suppose that whenever $x_n \in h_a \in \mathcal{H}$ and $x_\lambda \in [n]\backslash h_a$, then we have $h_a\backslash\{x_n\} \cup \{x_\lambda\} \in \mathcal{H}$ (in other words, $\mathcal{H}$ is stable under shifting x_n).

   We may assume that if $\mathcal{H} = \{h_1, \ldots, h_m\}$, then there is some m_0 such that $x_n \in h_a$ for all $a < m_0$, and that $x_n \notin h_a$ otherwise. If $m = 1$, then we have case 1, since we have $n = k$, and so $2k - l \geq n$ holds. Let $m > 1$. If $m_0 = 1$, then the problem holds by the inductive hypothesis. Suppose that $m_0 \geq 3$. Let $\mu < \nu < m_0$. Then $|h_\mu \cup h_\nu| \leq 2k - l < n$, and there exists a $\lambda \in [n]\backslash(h_\mu \cup h_\mu)$. Then let $f_\mu = h_\mu\backslash\{x_n\}$. Then $f_\mu \cup \{x_\lambda\} \in \mathcal{H}$, and $|f_\mu \cap f_\nu| = |(f_\mu \cup \{x_\lambda\}) \cap f_\nu| = |(f_\mu \cup \{x_\lambda\}) \cap h_\nu| \geq k$, and therefore $k-1 \geq l$. Let $\mathcal{F} = \{f_1, \ldots, f_{m_0-1}\}$. If $m_0 = 2$, then $x_n \notin h_2$, and $|h_1 \cap h_2| \leq k - 1$, so that $k - 1 \geq l$, and $s - 1 \geq 1$, since we have $s - 1 + l \geq k - 1$ and we have $s - 1 < k - 1$, and $k - 1 \geq l \geq 1$. Then we have

   $$m_0 \frac{\binom{2(k-1)-l}{s-1}}{\binom{2(k-1)-l}{k-1}} \leq |\delta_{s-1}(\mathcal{F})| = p,$$

   which holds by case 2 in the case $s = 2$, and when $s > 2$, we can use the inductive hypothesis. Let $\mathcal{F}' = \{h_{m_0}, \ldots, h_n\}$. Then if $k \leq n - 1$, we have

   $$(m - m_0) \frac{\binom{2k-l}{s}}{\binom{2k-l}{s}} \leq |\delta_s(\mathcal{F}')| = r,$$

   while if $k = n$, we have case 1, because $2k - l \geq n$. Then, we have (since $k > s$ and $s + l - k \geq 0$),

   $$\frac{\binom{2k-l}{s}}{\binom{2k-l}{k}} \leq \frac{\binom{2(k-1)-l}{s-1}}{\binom{2(k-1)-l}{k-1}}.$$

   Combining the previous three displays, we find

   $$m \frac{\binom{2k-l}{s}}{\binom{3k-l}{k}} \leq p + r.$$

   Denote by d_ν for $\nu < p$ the elements of $\delta_{s-1}(\mathcal{F})$, and by c_ν for $\nu < r$ the elements of $\delta_s(\mathcal{F}')$. Let $e_\nu = d_\nu + \{x_n\}$ for $\nu < p$. Then $|e_\nu| = g$, and

$e_\mu \neq e_\nu$. Moreover, we have that for every $\nu < p$, there exists an index $\mu < m_0$ such that $d_\nu \subset f_\mu$. Hence $e_\nu \subset h_\mu$. Then $e_\nu \in \delta_s(\mathcal{H})$, and we have that $e_\mu \neq c_\nu$. Then $c_1, \ldots, c_r, e_0, \ldots, e_p$ are distinct elements of $\delta_s\mathcal{H}$, so $p + r \leq |\delta_s(\mathcal{H})|$.

(b) Suppose that there are $h \in \mathcal{H}$ and $\lambda \in [n]\backslash h$ such that $n \in h$ and $h\backslash\{x_n\} \cup \lambda \notin \mathcal{H}$. Then $\lambda < n$. We may assume that the sets are labeled in such a way that the following hold:

$$\begin{array}{llll} x_n \in h_\nu, & x_\lambda \notin h_\nu & b_\nu = h_\nu\backslash\{x_n\} \cup \{\lambda\} \notin \mathcal{H} & (\nu < m_0) \\ x_n \in h_\nu, & x_\lambda \notin h_\nu & c_\nu = h_\nu\backslash\{x_n\} \cup \{\lambda\} \in \mathcal{H} & (m_0 \leq \nu < m_1) \\ x_n \in h_\nu, & x_\lambda \in h_\nu & (m_1 \leq \nu < m_2) & \\ x_n \notin h_\nu, & & (m_2 \leq \nu \leq m) & \end{array}$$

Set $b_\nu = h_\nu$ for $m_0 \leq \nu \leq m$. Then we must prove that $B = \{b_0, \ldots, b_n\}$ is l-intersecting, k-uniform, and on m vertices. We must prove that $b_\mu \neq b_\mu$ and $|b_\mu \cap b_\nu| \geq l$.

For $\mu < \nu < n_0$, or $n_0 \leq \mu \leq \nu$, this is clear. Let $\mu < m_0 \leq \nu$. Then $b_\mu \in \mathcal{H}$, and $b_\nu = a_\nu \in \mathcal{H}$, so $b_\mu \neq b_\mu$.

If $m_0 \leq \nu < m_1$, then $c_\nu \in \mathcal{H}$, and there are l distinct common elements of h_μ and c_ν. We have that x_λ and x_m are not among these, therefore they are common elements also of b_μ and b_ν.

If $m_1 \leq \nu < m_2$, then $|h_\mu \cap h_\nu| \geq l$, but $x_\lambda \notin |h_\mu \cap h_\nu|$. If instead of a_μ we take b_μ, then we lose at most one element from the intersection, but x_λ, which is a common element, is not among these, so $|b_\mu \cap b_\nu| \geq l$.

Finally, if $m_2 \leq \nu \leq m$, then h_μ and h_ν have k common elements. We have that x_n does not belong to them, so the same k elements are also common elements of b_μ and b_ν.

Now we must show that $|\delta_s(\mathcal{H})| \geq |\delta_s(B)|$. Let c be a set such that $|c| = s$, and $c \in \delta_s(B)$, but $c \notin \delta_s(\mathcal{H})$. Then $c \subset b_\nu$ for some $\nu \leq m$. Clearly, $\nu < m_0$. Then we have that $x_\lambda \in c$, because $c \not\subset h_\nu = b_\nu + \{x_n\} - \{x_\lambda\}$. On the other hand, we have that $x_n \notin c$, because $x_n \notin b_\nu$.

Let $d = c\backslash\{x_\lambda\} + \{x_n\}$. Then $d \subset h_\nu$, and $d \in \delta_s(\mathcal{H})$. However, $d \notin \delta_s(B)$, since if $d \subset b_\nu$ holds for some ν, then $m_0 \leq \nu < m_2$. If $\nu < n_1$, then $c \subset c_\nu = h_\nu\backslash\{x_n\} \cup \{x_\lambda\}$ holds, and since $c_\nu \in \mathcal{H}$, it follows that $c \in \delta_s(\mathcal{H})$, which contradicts our supposition. However, if $m_1 \leq \nu < n_2$, then $c \subset h_\nu$ because we have $x_\lambda \in b_\nu = a_\nu$, and $x_n \in b_\nu = a_\nu$, and this is also a contradiction.

Now we have associated a set d to every set c, where $c \in \delta_s(B)$, but $c \notin \delta_s(\mathcal{H})$ in such a way that d is an element of $\delta_s(\mathcal{H})$ but not an element of $\delta_s(B)$. It follows that $|\delta_s(\mathcal{H})| \geq |\delta_s(B)|$. Since for fixed m we supposed $\mathcal{H}$ to be the system for which $|\delta_s(\mathcal{H})|$ was minimal, equality must hold. However, we have

$$f(b_1, \ldots, b_m) - f(h_1, \ldots, h_m) = m_0(\lambda - n) < 0,$$

where $f(a_1, \ldots, a_m)$ sums the indices from each vertex of each edge. which contradicts the minimal-sum property of $\mathcal{H}$. Therefore, this case cannot occur.

□

Now, we will show that this theorem implies the Erdős-Ko-Rado theorem. Let $\mathcal{H} = \{h_1 \ldots, h_m\}$, and let $\mathcal{G} = \{[n]\backslash h_1, \ldots, [n]\backslash h_n\} = \{g_1, \ldots, g_n\}$. Then $|g_i| = n-k \geq k$, and $|g_i \cap g_j| = n - |h_i \cup h_j| \geq n-2k+1$. Now, apply the theorem on g with $s = k$ (which we can do since $1 \leq s = k \leq n-k$, $1 \leq n-2k+1 \leq n-k$, and $K + n - 2k + 1 \geq n-k$). Then

$$|\mathcal{H}|\frac{\binom{n-1}{k}}{\binom{n-1}{n-k}} \leq |\delta_k(\mathcal{G})|.$$

Let $c \in \delta_k(\mathcal{G})$. Then there exists a g_j such that $c \subset g_j$. Then $c \cap h_j = \emptyset$. Consequently, we find

$$|\delta_k(B)| + |\mathcal{H}| \leq \binom{n}{k}.$$

Applying the previous inequality, we find

$$\begin{aligned}
|\mathcal{H}| &\leq \frac{\binom{n}{k}}{\frac{\binom{n-1}{k}}{\binom{n-1}{n-k}} + 1} \\
&= \binom{n}{k} \frac{\frac{(n-1)!}{(n-k)!(k-1)!}}{\frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(n-k)!(k-1)!}} \\
&= \binom{n}{k} \frac{k!(n-1-k)!}{(k-1)!(k+n-k)(n-k-1)!} \\
&= \binom{n-1}{k-1}.
\end{aligned}$$

2.6 A New Short Proof: Frankl and Füredi

A new short proof of the Erdős-Ko-Rado Theorem was given in 2012 by Frankl and Füredi in [15]. First, we must recall some notation.

As before, let $\delta_s(\mathcal{H})$ denote the family of s-subsets of the edges of $\mathcal{H}$:

$$\delta_s(\mathcal{H}) = \{S \ : |S| = s, \ S \subset f \in \mathcal{H}\}.$$

If $\mathcal{H}$ is k-uniform, and l-intersecting, then it was shown by Katona (in [21], see Theorem 2.5.1 with $s = k - l$) that

$$|\mathcal{H}| \leq |\delta_{k-l}\mathcal{H}|. \tag{2.1}$$

Now we will begin the proof. Define the following sets:

$$\mathcal{H}_0 = \{f \in \mathcal{H} \mid x_1 \notin f\},$$
$$\mathcal{H}_1 = \{f \in \mathcal{H} \mid x_1 \in f\},$$
$$\mathcal{G}_1 = \{f \backslash \{x_1\} \mid x_1 \in f \in \mathcal{H}\},$$
$$\mathcal{G}_0 = \{\{x_2, \ldots, x_n\} \backslash f \mid f \in \mathcal{H}_0\}.$$

To see examples of these sets, refer to Figures 2.1 and 2.2.

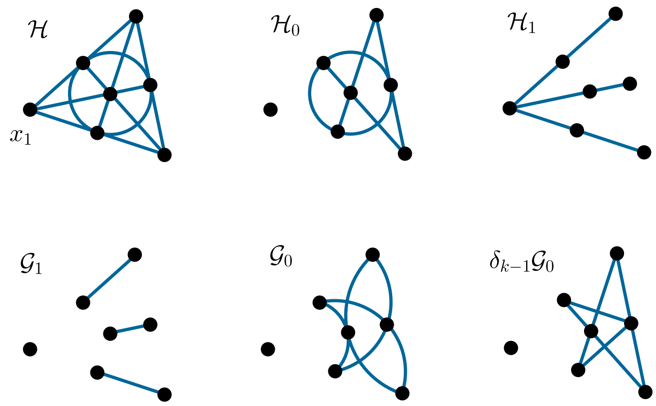


Figure 2.1: The graphs $\mathcal{H}_1, \mathcal{H}_0, \mathcal{G}_1, \mathcal{G}_0$, and $\delta_{k-1}\mathcal{G}_0$ for $\mathcal{H}$ is the Fano plane.

Then $\mathcal{G}_0$ is $(n-1-k)$-uniform. Since $\mathcal{H}$ is intersecting, we have that if $h_1 \in \mathcal{H}_1$, then

$$h_1 \backslash \{x_1\} = g_1 \in \mathcal{G}_1$$

is not contained in any member of $\mathcal{G}_0$. Then we find that

$$\mathcal{G}_1 \cap \delta_{k-1}\mathcal{G}_0 = \emptyset.$$

We have that $\mathcal{G}_0$ is $n-2k$ intersecting, since if $g, g' \in \mathcal{G}_0$, then

$$|g \cap g'| = |(\{x_2, \ldots, x_n\} \backslash f) \cap (\{x_2, \ldots, x_n\} \backslash f')| = (n-1) - 2k + |f \cap f'| \geq n - 2k.$$

Then by Equation 2.1, we find that $|\mathcal{G}_0| \leq |\delta_{k-1}\mathcal{G}_0|$, since $\mathcal{G}_0$ is $(n-k-1)$-uniform, and $(n-2k)$-intersecting. Then,

$$|\mathcal{H}| = |\mathcal{H}_1| + |\mathcal{H}_0| = |\mathcal{G}_1| + |\mathcal{G}_0| \leq |\mathcal{G}_1| + |\delta_{k-1}(\mathcal{G}_0)| \leq \binom{n-1}{k-1},$$

since $\mathcal{G}_1$ and $\delta_{k-1}\mathcal{G}_0$ are disjoint $k-1$ uniform hypergraphs on $n-1$ vertices.

Equality holds in Equation 2.1 when $n = k$, $\mathcal{H} = \emptyset$, or $\mathcal{H} = \binom{2n-k}{n}$. So, for $n > 2k$, equality implies that either $\mathcal{G}_0 = \emptyset$ so that $x_1 \in \cap\mathcal{H}$, or $\mathcal{G}_0 = K_{n-2}^{n-k-1}$ and so there is an $x \in X$ such that $x \in \cap\mathcal{H}$.

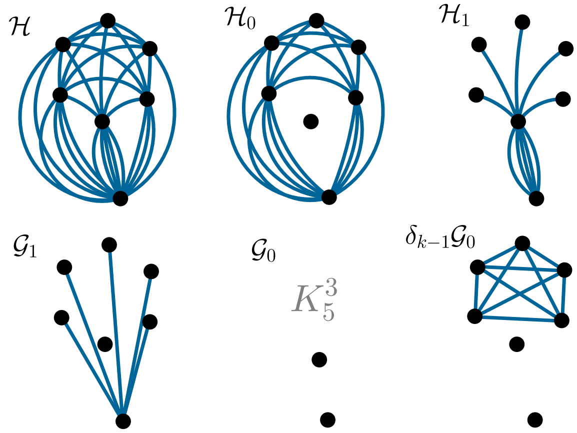


Figure 2.2: The graphs $\mathcal{H}_1, \mathcal{H}_0, \mathcal{G}_1, \mathcal{G}_0$, and $\delta_{k-1}\mathcal{G}_0$ for the hypergraph $\mathcal{H} = \{e \in K_n^k \mid x_2 \in e\}$.

2.7 Linearly Independent Polynomials

In [17], the authors give a proof of the Erdős-Ko-Rado Theorem which uses the method of linearly independent polynomials.

Consider a hypergraph $\mathcal{H} = \{e_1, \dots, e_m\}$, where $e_i \subset \{x_1, \dots, x_n\} = X$. For a set $e \subset X$, and some set $P \subset X$, we will say that the set e *satisfies the intersection property* (P, α) if we have that $|e \cap P| = \alpha$. Suppose that for each $e_i \in \mathcal{H}$, a collection of at most s intersection properties are given:

$$R_i = \{(P_{i1}, \alpha_{i1}), \dots, (P_{is}, \alpha_{is})\}.$$

Lemma 2.7.1. *Suppose that for each $e_i \in \mathcal{H}$, one can find $X_i \subset \{x_1, \dots, x_n\}$ such that*

1. *X_i does not satisfy any condition in R_i, and*

2. *X_i satisfies at least one condition for each R_j where $j > i$.*

Then if $m = |\mathcal{H}|$, we have

$$m \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}.$$

Proof. Define the n-variable real polynomial f_i as follows:

$$f_i(x_1, \dots, x_n) = \prod_{1 \le u \le s} \left(\left(\sum_{v \in P_{iu}} x_v \right) - \alpha_{iu} \right).$$

Let $\hat{x}$ denote the characteristic vector of $x \subset \{x_1, \dots, x_n\}$. Then the dot product $\hat{x} \cdot \hat{y} = |x \cap y|$ for all $x, y \subset [n]$. Then we find that

$$f_i(\hat{x}) = \prod_u (\hat{x}\hat{P_{iu}} - \alpha_{iu}) = \prod_u (|x \cap P_{iu}| - \alpha_{iu}).$$

Then the conditions of the lemma imply that $f_j(\hat{x}_i) \neq 0$ when $i = j$, and $f_j(\hat{x}_i) = 0$ when $i < j$. Now, we will define the integer coefficient n-variable multilinear polynomial g_i by multiplying out all factors of f_i, and replacing higher order factors x_v^l with simply x_v. Then for a vector x containing only 0s and 1s, we have $f_i(x) = g_i(x)$, so that $g_i(\hat{x}_i) \neq 0$ and $g_j(\hat{x}_i) = 0$ for $i < j$.

The multilinear n-variable real polynomials of degree less than or equal to s form a vector space of dimension $\binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}$ over the reals. We claim that the g_i are linearly dependent. Suppose otherwise; then there is a linear dependency such that

$$c_1 g_1(x) + \cdots + c_s g_s(x) = 0$$

holds for all $x \in \mathbb{R}^n$. Suppose i is the smallest integer with $c_i \neq 0$. Then substitute $\hat{x}_i$ in to the equation above. Then by the observations from before, we find that $c_i g_i(\hat{x}_i) = 0$, a contradiction. Therefore the g_i are linearly independent, and so we have that

$$s \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0},$$

as desired. □

Proof of Erdős-Ko-Rado. Let $\mathcal{H}$ be a k-uniform, intersecting hypergraph on the vertex set $X = \{x_1, \ldots, x_n\}$, where $n \geq 2k$. Let $|\mathcal{H}| = m$, and define $s = k - 1$. To use the lemma, we will add to $\mathcal{H}$ another

$$\binom{n}{k-1} + \binom{n}{k-2} + \cdots + \binom{n}{0} - \binom{n-1}{k-1} = 2 \times \left(\sum_{k-2 \geq u \geq 0} \binom{n-1}{u} \right) \tag{2.2}$$

sets and conditions. Let $p \in [n]$. Define

$$\begin{aligned}
\mathcal{H}_0 &= \{h \in \mathcal{H} \mid x_p \notin h\}, \\
G_0 &= \{h \subset X \mid x_p \notin h,\ 0 \leq |h| \leq k-2\}, \\
\mathcal{H}_1 &= \{h \in \mathcal{H} \mid x_p \in h\}, \\
G_1 &= \{h \subset X \mid x_p \in h,\ 1 \leq |h| \leq k-1\}.
\end{aligned}$$

Let $A = \mathcal{H}_0 \cup G_0 \cup \mathcal{H}_1 \cup G$. Order the elements of A as follows: first, put the members of $\mathcal{H}_0$ in an arbitrary order. Then, put the elements of G_0 in order of increasing size, then put the elements of $\mathcal{H}_1$ in an arbitrary order, and then finally put the elements of G_1 in order of increasing size. Then for $a_i \in A$, associate $x_i \subset X$ and at most $k-1$ intersection conditions:

1. For $h \in \mathcal{H}_0$, let $x = X \backslash \{x_p\} \backslash h$, with intersection conditions (h, α) for $1 \leq \alpha \leq k-1$.

2. For $h \in G_0$, let $x = h$, with intersection conditions $(\{x_i\}, 0)$ for each $x_i \in h$.

3. For $h \in \mathcal{H}_1$, let $x = h \backslash \{x_p\}$, with intersection conditions $(h \backslash \{x_p\}, \alpha)$ for $1 \leq \alpha \leq k-2$.

4. For $h \in G_1$, let $x = h$, with intersection conditions $(\{x_i\}, 0)$ for each $x_i \in h$.

It is straightforward (but somewhat tedious) to check that the $\{e_i, x_i, (P_{iu}, \alpha_{iu})\}$ defined satisfy the conditions of Lemma 2.7.1. Then

$$|\mathcal{H}| + |G_0| + |G_1| \leq \binom{n}{k-1} + \binom{n}{k-2} + \ldots + \binom{n}{0}.$$

Since $|G_0| = |G_1| = \sum_{k-2 \geq u \geq 0} \binom{n-1}{u}$, we have the theorem by Equation 2.2. $\square$

Chapter 3

Extensions of the Erdős-Ko-Rado Theorem

There have been many extensions and generalizations of the Erdős-Ko-Rado Theorem. Deza and Frankl discuss many of them in [7]. Here, we provide a study of extremal sets under the Erdős-Ko-Rado conditions, a rephrasing of the Erdős-Ko-Rado theorem in terms of independent sets and an exploration of higher orders of independence, a discussion of Sperner sets, and a lower bound on the chromatic number of the Kneser graph. First, we recall the following definition.

Definition 3.0.2. A hypergraph $\mathcal{H} \subset \mathcal{P}(X)$ is l-intersecting if for any $f, f' \in \mathcal{H}$, we have that $|f \cap f'| \geq l$.

The original Erdős-Ko-Rado Theorem from [10] concerns l-intersecting, uniform hypergraphs. For this thesis, we have focused on the $l = 1$ case, but the more general result is as follows.

Theorem 3.0.3. *Let $n > k > l > 0$, and suppose $\mathcal{H}$ is a k-uniform, l-intersecting hypergraph on n vertices. Then for $n > n_0(k, l)$, we have that*

$$|\mathcal{H}| \leq \binom{n-l}{k-l},$$

and equality holds if and only if there is some l-subset of X which is contained in every edge.

The original proof by Erdős, Ko, and Rado involves the exchange operation. Attempts have been made to find a Katona-type proof (see Section 2.2) for this generalization, but they are without success. [20]

3.1 Extremal Sets

An *extremal set* is a hypergraph which exhibits the upper or lower bound for some statistic. In Theorem 3.0.3, the extremal sets are the hypergraphs $\mathcal{H}$ where

$$\left|\bigcap_{e \in \mathcal{H}} e\right| = l,$$

meaning that there is a set A of size l such that $A \subset e$ for all $e \in \mathcal{H}$.

We may wonder what happens if we disallow these cases. What are the new extremal sets, and what is their maximal size? Hilton and Milner proved the following in [19].

Theorem 3.1.1. *If, in addition to the usual assumptions of the 1-intersecting Erdős-Ko-Rado theorem, we have*

$$\bigcap_{e \in \mathcal{H}} e = \emptyset,$$

then we find

$$|\mathcal{H}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

In this case, equality holds if and only if for some $x \in X$, and some $D \subset X$, $|D| = k$, where $x \notin D$, we have

$$\mathcal{H} = \{f \subset X \mid |f| = k,\ x \in f,\ f \cap D \neq \emptyset\} \cup \{D\}.$$

Erdős, Rothschild, and Szemerédi asked a similar question (unpublished). Let $c \in \mathbb{R}$, with $0 < c < 1$. How large can an intersecting k-uniform hypergraph be if no vertex has degree greater than $c|\mathcal{H}|$? In the case where $c = 2/3$, they proved

$$|\mathcal{H}| \leq |\mathcal{H}_{2,3} = \{F \subset X \mid |F| = k,\ |F \cap \{x_1, x_2, x_3\}| \geq 2\}|.$$

In [14], Frankl extends these results for l-intersecting hypergraphs.

Theorem 3.1.2 (Frankl, [14])**.** *Let $\mathcal{H}$ be a l-intersecting, k-uniform hypergraph. Suppose that $|\cap \mathcal{H}| < l$, and that $|\mathcal{H}|$ is maximal subject to these constraints. Then for $n > n_0(k)$, we have*

(a) When $k > 2l + 1$ or $k = 3, l = 1$:

There exist $D_1, D_2 \subset X$ where $D_1 \cap D_2 = \emptyset$, and $|D_1| = l$, $|D_2| = k - l + 1$ such that

$$\begin{aligned}\mathcal{H} =& \{f \subset X \mid |f| = k, f \cap D_2 \neq \emptyset, D_1 \subset f\} \\ & \cup \{f \subset X \mid |f| = k, D_2 \subset f, |f \cap D_1| = l - 1\}.\end{aligned}$$

(b) When $k \leq 2l + 1$:

There exists $D \subset X$, where $|D| = l + 2$ such that

$$\mathcal{H} = \{f \subset X \mid |f| = k, |f \cap D| \geq l + 1\}.$$

3.2 Independence

Another extension of the Erdős-Ko-Rado theorem which was first explored by Erdős in [8] concerns independent sets. He asks, given an r-uniform hypergraph on n vertices, how many edges must the hypergraph have in order to guarantee that there is an independent set of size k?

Definition 3.2.1. Let $\mathcal{H}$ be a hypergraph. We will say that a set of edges E is *independent* if for any $e, e' \in E$, we have that $e \cap e' = \emptyset$.

Let $f(n; r, k)$ denote the smallest integer such that every r-uniform hypergraph on n vertices with $f(n; r, k)$ edges has k independent edges. In this context, the Erdős-Ko-Rado theorem says that

$$f(n; r, 2) = \binom{n-1}{r-1} + 1.$$

Let $g(n; r, k-1)$ denote the number of r-edges from $x_1, \ldots, x_n$, each of which contain at least one element from $x_1, \ldots, x_{k-1}$. Then we obtain a lower bound for f:

$$f(n; r, k) > g(n; r, k-1),$$

since the hypergraph described by g does not contain k independent sets (it contains $k-1$). Counting the edges in the graph, we find

$$g(n; r, k-1) = \sum_{i=1}^{\min(r,k-1)} \binom{k-1}{i}\binom{n-k+1}{r-i} \geq (k-1)\binom{n-k+1}{r-1}. \tag{3.1}$$

The following result was proved about the relationship between $f(n; r, k)$ and $g(n; r, k)$.

Theorem 3.2.2 (Erdős [8]). *For $n > c_r k$ (c_r is a constant depending only on r),*

$$f(n; r, k) = g(n; r, k-1) + 1.$$

Proof. The proof is by induction on k. For $k = 2$, the result is the Erdős-Ko-Rado theorem:

$$g(n; r, 2) = \sum_{i=1}^{1} \binom{2-1}{i}\binom{n-2+i}{r-i} = \binom{n-1}{r-1}.$$

Now assume the theorem holds for $k-1$, and prove it for k. Let $n > c_r k$ and consider an r-uniform hypergraph $\mathcal{H}$ on n vertices with $1 + g(n; r, k-1)$ edges. Without loss of generality, assume that $\max(v(x_i)) = v(x_1)$. We distinguish two cases.

1. Assume $v(x_1) < \frac{1+g(n;r,k-1)}{(k-1)r}$. Let $\{R_1, \ldots, R_l\}$ be a maximal system of independent edges. We will show that $l \geq k$.

   If $l < k$, then $R_1, \ldots, R_l$ contain at most $lr \leq (k-1)r$ vertices. Since $\{R_1, \ldots, R_l\}$ is a maximal system of independent edges, we have that the number of edges containing any of these vertices is less than $1 + g(n; r, k)$. Since $\mathcal{H}$ has $1 + g(n; r, k)$ edges, this means that there is at least one edge R_{l+1} which is independent from the $R_1, \ldots, R_l$. This contradicts maximality. Therefore, $l \geq k$, completing the first case.

2. Assume $v(x_1) \geq \frac{1+g(n;r,k-1)}{(k-1)r}$. Now consider the r-graph $\mathcal{H}_1$ with vertices $x_1, \ldots, x_n$ and all edges from $\mathcal{H}$ which did not contain x_1. By noting that

$$g(n;r,k-1) = \binom{n}{r} - \binom{n-k+1}{r},$$

we see that the number of edges in $\mathcal{H}_1$ is at least

$$\begin{aligned} 1 + g(n;r,k-1) - \binom{n-1}{r-1} &= 1 + \binom{n}{r} - \binom{n-k+1}{r} - \binom{n-1}{r-1} \\ &= 1 + \binom{n-1}{r} - \binom{n-k+1}{r} \\ &= 1 + g(n;r,k-2), \end{aligned}$$

since at most $\binom{n-1}{r-1}$ edges contain x_1. Then by the induction hypothesis, $\mathcal{H}_1$ contains at least $k-1$ independent edges. Now, we must show that there is an edge in $\mathcal{H}$ containing x_1 which has none of the other $(k-1)r$ vertices of $R_1, \ldots, R_{k-1}$.

Observe that the number of edges containing x_1 and x_i is at most $\binom{n-2}{r-2}$, so the number of edges containing x_1 and one vertex from the $R_1, \ldots, R_{k-1}$ is at most

$$(k-1)r\binom{n-2}{r-2}.$$

By the assumption and Equation 3.1, we have that for for $n > c_r k$,

$$(k-1)r\binom{n-2}{r-2} < v(x_1).$$

Hence, there is an edge which is independent from the $R_1, \ldots, R_{k-1}$.

□

We obtain the following corollary from Theorem 3.2.2.

Corollary 3.2.3. *Let $s \geq 2$, and let $\mathcal{H}$ be a k-uniform hypergraph such that $\mathcal{H}$ does not contain $e_1, \ldots, e_s$ such that $e_i \cap e_j = \emptyset$ for all $i, j \in \{1, \ldots, s\}$. Then for $n > n_0(k,s)$, we have*

$$|\mathcal{H}| \leq \binom{n}{k} - \binom{n-s+1}{k}.$$

3.3 Sperner Sets

Before we discuss Sperner sets, we will give a slight improvement of the Erdős-Ko-Rado theorem. Recall that in the Erdős-Ko-Rado theorem, we required $2k \leq n$. The following result gives conditions under which we can use smaller n.

Lemma 3.3.1 (Frankl, [13]). *Let X be finite, and $|X| = n$. Let $\mathcal{H}$ be a k-uniform hypergraph. Let $t > 1$ such that $\frac{tk}{t-1} \leq n$. If, for any edges $h_1, \ldots, h_t \in \mathcal{H}$, we have that*

$$\bigcap_{i=1}^{t} h_i \neq \emptyset,$$

then

$$|\mathcal{H}| \leq \binom{n-1}{k-1}.$$

The Erdős-Ko-Rado theorem implies this result for $2k \leq n$, so all we have done is lowered the bound on n for $t > 2$. When $t = 2$, this is the Erdős-Ko-Rado theorem.

Proof. This proof will be similar to the proof of the Erdős-Ko-Rado theorem by Katona using cyclic permutations. Let $\mathcal{H}^c = \{f \mid X \backslash f \in \mathcal{H}\}$. Then every member of $\mathcal{H}^c$ has cardinality $n-k$, and $n-k \geq n/t$, since

$$n \geq \frac{tk}{t-1} \implies \frac{tn-n}{t} \geq k \implies n - \frac{n}{t} \geq k \implies n-k \geq \frac{n}{t}.$$

By the intersecting property, we find that for any $h_1, \ldots, h_t \in \mathcal{H}^c$, we have that

$$\bigcup_{j=1}^{t} h_j \neq X.$$

Let $(x_1 \cdots x_n) = \pi$ be a cyclic permutation of the elements of X. For an edge $h \in \mathcal{H}$, we will write $h \subset \pi$ if the vertices of h are consecutive elements of π (like before, in Katona's proof, we say that π extends h). If there exists an $h' \in \mathcal{H}$ such that $h' \subset \pi$, without loss of generality we may assume that the last element of h' is x_n.

For every $h \subset \pi$, make the pair (h, j) where x_j is the last element of h under the ordering π. To h', we make the pairs (h', j) for all $n \leq j \leq t(n-k)$. Suppose r edges are subsets of π. Then we have formed $r + t(n-k) - n$ pairs, whose second entry is a unique element from the interval $[1, t(n-k)]$. Consider these indices mod $n-k$ by letting

$$\mathcal{H}_r = \{(n, j) \mid h \subset \pi,\ j \equiv r \mod n-k\}.$$

Since edges are not duplicated, we know $|\mathcal{H}_r| \leq t$. However, if $|\mathcal{H}_r| = t$, then $\cup\mathcal{H}_r = X$, a contradiction. Therefore, there exists an x_j such that $j \equiv r \mod n-k$ for each possible r (of which there are $n-k$). Hence,

$$(r + t(n-k) - n) + (n-k) \leq t(n-k),$$

so $r \leq k$. Then, as in the proof of the Erdős-Ko-Rado by Katona, we have

$$\underbrace{|\mathcal{H}|k!(n-k)!}_{\#\text{ cyclic extensions}} \leq \underbrace{k(n-1)!}_{\#\text{ possible cyclic extensions}}$$

and so the theorem is proved. □

Now, we can expand this result by removing the constraint that $\mathcal{H}$ is k-uniform. Instead, we will assume that $\mathcal{H}$ is *Sperner*.

Definition 3.3.2. A hypergraph $\mathcal{H}$ is *Sperner* if for any distinct $h, h' \in \mathcal{H}$, we have $h \not\subset h'$.

Frankl gives the following in [13], in which he expands Lemma 3.3.1 for Sperner hypergraphs.

Theorem 3.3.3 (Frankl, [13]). *Let $|X| = n$, and let $\mathcal{H}$ be Sperner with edges no larger than k. Let the sets in $\mathcal{H}$ be t-wise nondisjoint, and suppose that $tk \leq (t-1)n$. Then*

$$\sum_{j=1}^{k} \frac{|\mathcal{H}_j|}{\binom{n-1}{j-1}} \leq 1,$$

where $\mathcal{H}_j = \{f \in \mathcal{H} \mid |f| = j\}$.

Note that for n, t, k as in the theorem, we have $\frac{n}{2}\binom{n-1}{k-1}$ increases monotonically, so that

$$1 \geq \sum_{j=1}^{k} \frac{|\mathcal{H}_j|}{\binom{n-1}{j-1}} \geq \frac{\sum_{j=1}^{k} |\mathcal{H}_j|}{\binom{n-1}{k-1}} = \frac{|\mathcal{H}|}{\binom{n-1}{k-1}},$$

so this theorem improves the Erdős-Ko-Rado theorem.

Proof. The proof requires the following inequality, which comes from the Kruskal-Katona theorem. Let $\mathcal{H}$ be a k-uniform hypergraph on X. If

$$|\mathcal{H}| \leq \binom{s}{k},$$

then

$$|\mathcal{H}'| \geq |\mathcal{H}| \frac{\binom{s}{k-1}}{\binom{s}{k}},$$

where $\mathcal{H}'$ is all $k-1$ sets which are subsets of edges of $\mathcal{H}$. Now, we proceed with the proof.

We apply induction to the number of nonzero $|\mathcal{H}_j|$'s. The base case is handled by Lemma 3.3.1, so we proceed to the induction step.

Let p be the smallest, and r the second smallest, index such that $|\mathcal{H}_j| \neq 0$. Let

$$B_r = \{b \subset X \mid \exists h \in \mathcal{H}_p,\ h \subset b,\ |b| = r\}.$$

Using the Kruskal-Katona theorem (r-p) times, we find

$$|B_r| \geq |\mathcal{H}_p| \frac{\binom{n-1}{r-1}}{\binom{n-1}{p-1}}.$$

The family $\bar{\mathcal{H}} = \mathcal{H}\backslash\mathcal{H}_p \cup B_r$ also satisfies the assumptions of the theorem, and the number of nonzero $|H_j|$'s has decreased by 1, so by the inductive hypothesis and the previous inequality, we find

$$\sum_{j=1}^{k} \frac{|\mathcal{H}_j|}{\binom{n-1}{j-1}} \leq \sum_{j=1}^{k} \frac{|\bar{\mathcal{H}}_j|}{\binom{n-1}{j-1}} \leq 1.$$

□

3.4 Chromatic Number of the Kneser Graph

Here we present an application of the Erdős-Ko-Rado theorem, in which we obtain a lower bound for the chromatic number of the Kneser graph.

Definition 3.4.1. The *chromatic number* of a graph G is the smallest number of colors needed to color the vertices of G such that there is no monochromatic edge.

Definition 3.4.2. The *Kneser Graph* $K(n, k)$ is the graph whose vertices are the k-element subsets of the numbers $\{1, \ldots, n\}$, and two vertices share an edge when the corresponding sets are disjoint.

Definition 3.4.3. An *independent set* W of a graph G is a subset of the vertices that pairwise share no edges.

Proposition 3.4.4. *The chromatic number of $K(n, k)$ is at least n/k.*

Proof. It is clear that if we have a coloring of a graph G, then each of the color classes is an independent set. By the Erdős-Ko-Rado theorem, the maximal size of an independent set in $K(n, k)$ is $\binom{n-1}{k-1}$, and so in a proper coloring of $K(n, k)$, there must be at least

$$\frac{\binom{n}{k}}{\binom{n-1}{k-1}} = \frac{n}{k}$$

colors. Hence, the chromatic number of $K(n, k)$ is at least n/k. □

It turns out that the chromatic number of the Kneser graph is actually $n - 2k + 2$. Kneser conjectured this in 1955, and it was proved by Lovász in 1978 using topological methods [29]. Other proofs have also been given each by Bárány (1978), Greene (2002), and Matoušek (2004).

Using the Baranyai theorem, we can actually determine something called the clique covering number of the Kneser graph. A *clique* of a graph is a complete subgraph. The *clique covering number* of a graph is the minimum number of cliques required to cover the vertex set of the graph. Then the Baranyai theorem implies that the clique covering number of $K(n, k)$ when k divides n is

$$\binom{n-1}{k-1},$$

since each factor in a factorization of K_n^k will represent a clique in $K(n, k)$.

Chapter 4

Proof of the Baranyai Theorem

This chapter contains the original proof of the Baranyai Theorem given by Baranyai in [3]. The proof involves several "integer making lemmas," and a complicated induction step which together yield a slightly more powerful but abstruse result. From there, the Baranyai theorem and typical generalizations quickly follow. We will start by stating the main theorem, and presenting the Baranyai theorem and other corollaries. Afterwards, we will proceed with proving the integer making lemmas and the induction step. Recall that an *almost regular* hypergraph is one where the degrees of any two vertices differ by at most one.

Theorem 4.0.5. *Let X be a set of n elements. Let A be an integer matrix with entries (a_{ij}), for $i \in [p]$ and $j \in [s]$, and let $h_1, \ldots, h_p$ be integers. If (a_{ij}) and $h_1, \ldots, h_p$ satisfy*

(1) $0 \leq h_i \leq n$

(2) $a_{ij} \geq 0$,

(3) $\sum_j a_{ij} = \binom{n}{h_i}$

then there exist $E_{ij}^{\nu} \subset X$ for $\nu \in [a_{ij}]$, $i \in [p]$, and $j \in [s]$, such that

(a) $|E_{ij}^{\nu}| = h_i$

(b) If $j_1 \neq j_2$ or $\nu_1 \neq \nu_2$, then $E_{ij_1}^{\nu_1} \neq E_{ij_2}^{\nu_2}$

(c) At any fixed j, the collection of E_{ij}^{ν} form an almost regular hypergraph on X.

Let us explain this result. Intuitively, the sets E_{ij}^{ν} are the edges of the hypergraph, and for a fixed i, j, the set $\{E_{ij}^{\nu} \mid \nu \in [a_{ij}]\}$ is a factor of the factorization, and the entire factorization is given by the ith row. So, each a_{ij} is the size of the factor, and we think of the E_{ij}^{ν} as "belonging" to a_{ij}. Then we may interpret the above as:

(a) tells us that the factors belonging to the ith row each have h_i elements

(b) tells us that the factors are disjoint, so that as a consequence of (3), all h_i-element subsets belong to some cell of the ith row

(c) tells us that the edges belonging to the same column form an almost regular hypergraph.

The Baranyai theorem and the other corollaries which we deduce from this theorem will only concern the case when $p = 1$.

4.1 Results

First, we will give some of the outcomes of Theorem 4.0.5. We prove a secondary result, from which the Baranyai theorem and its extensions readily follow. Let $N = \binom{n}{h}$.

Theorem 4.1.1. *Let $a_1, \ldots, a_s$ be numbers such that $\sum_{j=1}^{s} a_j = N$. Then the edges of K_n^h can be partitioned in almost regular hyper graphs $\mathcal{H}_j$ so that $|\mathcal{H}_j| = a_j$ for all $j \in [s]$.*

Proof. Apply Theorem 4.0.5 with $p = 1$ and $h_1 = h$. □

Note that either $v_{\mathcal{H}_j} = \lfloor \frac{a_j h}{n} \rfloor$ or $v_{\mathcal{H}_j} = \lceil \frac{a_j h}{n} \rceil$.

Corollary 4.1.2 (Baranyai theorems)**.**

1. *If $h \mid n$ then K_n^h is 1-factorizable.*

   Proof. Use Theorem 4.1.1 with $s = \binom{n-1}{h-1}$, and $a_j = n/h$. Then $v_{\mathcal{H}_j} = \frac{a_j h}{n} = 1$, so the theorem is proved. □

2. *K_n^h is d-factorizable if and only if $h \mid dn$ and $\frac{dn}{h} \mid N$.*

   Proof. Clearly, if K_n^h is d-factorizable then $h \mid dn$ and $\frac{dn}{h} \mid N$. Conversely, apply Theorem 4.1.1 with $s = \frac{Nh}{dn}$ and $a_j = \frac{dn}{h}$. □

3. *K_n^h is $\frac{h}{(n,h)}$-factorizable.*

   Proof. By the previous, is is enough to show that $\frac{n}{(n,h)} \mid \binom{n}{h}$. Observe that

$$\frac{h}{(h,n)}\binom{n}{h} = \frac{n}{(n,h)}\binom{n-1}{h-1}.$$

   Since $(\frac{n}{(n,h)}, \frac{h}{(n,h)}) = 1$, we have that $\frac{n}{(n,h)} \mid \binom{n}{h}$. □

4.2 Proof of the Baranyai Theorem

Now, we prove the Baranyai theorem. First, there are several lemmas which must be proved, and then a somewhat complicated induction step.

4.2.1 Integer Making Lemmas

Lemma 4.2.1. *Let $A, n \in \mathbb{Z}$, and $n \neq 1$. Then*

$$\left\lfloor \frac{A}{n} \right\rfloor = \left\lfloor \frac{A - \lceil A/n \rceil}{n-1} \right\rfloor, \qquad \text{and} \qquad \left\lceil \frac{A}{n} \right\rceil = \left\lceil \frac{A - \lfloor A/n \rfloor}{n-1} \right\rceil.$$

Proof. Let a, q be such that $A = an + q$, with $0 \leq q < n$. Then $a = \lfloor A/n \rfloor$. Then if $q \neq 0$, we have

$$\left\lfloor \frac{A - \lceil A/n \rceil}{n-1} \right\rfloor = \left\lfloor \frac{an + q - a - 1}{n-1} \right\rfloor = \left\lfloor a + \frac{q-1}{n-1} \right\rfloor = \left\lfloor \frac{A}{n} \right\rfloor.$$

If $q = 0$, then

$$\left\lfloor \frac{A - \lceil A/n \rceil}{n-1} \right\rfloor = \left\lfloor \frac{an - a}{n-1} \right\rfloor = a = \left\lfloor \frac{A}{n} \right\rfloor.$$

The other equality follows similarly. □

For the remainder of the chapter, Let $\mathcal{H}$ be a hypergraph on X, where $\mathcal{H} = \{E_1, \ldots, E_N\}$ (for this chapter only we will use capital letters to denote edges). Let $n = |X|$, and let $x \in X$. Denote by $\mathcal{H}\backslash x$ the hypergraph with vertex set $X\backslash x$ and edge set $\{E_1\backslash x, \ldots, E_N\backslash x\}$. We note that if $\mathcal{H}$ is almost regular, then so is $\mathcal{H}\backslash x$.

If l is an integer and d is a real number, let $l \approx d$ mean that either $\lfloor d \rfloor = l$ or $\lceil d \rceil = l$ holds. If $\mathcal{H}$ is almost regular, then

$$v_{\mathcal{H}}(x) \approx \frac{\sum_{i=1}^{N} |E_i|}{n},$$

because the right hand side is the average degree of a vertex in $\mathcal{H}$.

Lemma 4.2.2. *If $\mathcal{H}\backslash x$ is almost regular and*

$$v_{\mathcal{H}}(x) \approx \frac{\sum_{i=1}^{N} |E_i|}{n},$$

then $\mathcal{H}$ is almost regular.

Proof. We only need to show that for any $y \in X$, we have that $|v_{\mathcal{H}}(x) - v_{\mathcal{H}}(y)| \leq 1$. Let $A = \sum_{i=1}^{N} |E_i|$. Then $v_{\mathcal{H}}(x) \approx A/n$. Clearly,

$$v_{\mathcal{H}}(y) = v_{\mathcal{H}\backslash x}(y) \approx \frac{A - v_{\mathcal{H}}(x)}{n-1}.$$

Then by Lemma 4.2.1, we have that $v_{\mathcal{H}}(y) \approx A/n$, so we are done. □

Now, we will have a seemingly unrelated lemma about matrices, which will prove to be very useful later in the induction step proving Theorem 4.0.5.

Lemma 4.2.3. *Let (ϵ_{ij}) where $i \in [p]$ and $j \in [s]$ be a matrix of real numbers. Then there exists an integer matrix matrix (e_{ij}) such that* *(a)* $e_{ij} \approx \epsilon_{ij}$

(b) $\sum_i e_{ij} \approx \sum_i \epsilon_{ij}$

(c) $\sum_j e_{ij} \approx \sum_j \epsilon_{ij}$

(d) $\sum_i \sum_j e_{ij} \approx \sum_i \sum_j \epsilon_{ij}$.

Proof. First, we will show that we may assume that every row and column sum is 0. Set

$$\beta_i = \sum_{j=1}^{s} \epsilon_{ij} \qquad \gamma_j = \sum_{i=1}^{p} \epsilon_{ij}.$$

Then add another row and column to the matrix so that $\epsilon_{p+1,j} = -\gamma_j$, $\epsilon_{i,s+1} = -\beta_i$, and $\epsilon_{p+1,s+1} = \sum_{i=1}^{p} \beta_i$. Then in the new matrix, all rows and columns sum to 0. Also, if (a_{ij}) is a $p+1$ by $s+1$ matrix satisfying the conclusions of the theorem then the submatrix $\{a_{ij} \mid 1 \le i \le p,\ 1 \le j \le s\}$ will work for the original (ϵ_{ij}), since for $i \in [p]$,

$$\sum_{j=1}^{s} a_{ij} = -a_{i,s+1} \approx -(-\beta_i) = \beta_i = \sum_{j=1}^{s} \epsilon_{ij}.$$

Similarly, this works for the column sums and for the total sum.

Now, suppose that in the matrix (ϵ_{ij}), every row and column sum is 0. Choose a matrix $A = (e_{ij})$ with all row and column sums 0 such that $\lfloor \epsilon_{ij} \rfloor \le e_{ij} \le \lceil \epsilon_{ij} \rceil$ for all entries, and maximize the number of integer entries. Suppose for contradiction that not every entry in the matrix is an integer.

Call a sequence of entries $a_0, a_1, \ldots, a_{2t} = a_0$ a *circuit* if no a_i is an integer, and we have that a_0 and a_1 are in the same row, a_1 and a_2 are in the same column, and so on. Note that no row or column contains exactly one non-integer value, since the rows and columns sum to 0. Therefore, A contains a circuit. Given a circuit $a_0, a_1, \ldots, a_{2t} = a_0$, let $\epsilon = \min\{\lceil a_k \rceil - a_k, a_k - \lfloor a_k \rfloor\}$. Suppose that $\epsilon = \lceil a_k \rceil - a_k$ for some $k \in [2t]$. Then replace a_k with $a_k + \epsilon$, a_{k+1} with $a_{k+1} - \epsilon$, and so on around the circuit. The new matrix satisfies $\lfloor \epsilon_{ij} \rfloor \le e_{ij} \le \lceil \epsilon_{ij} \rceil$, and all row and column sums are 0, and it has more integer entries than A. This contradicts our choice of A. $\square$

4.2.2 The Induction Step

We will induct over n, the size of the vertex set. If $n = 1$, then the theorem holds.

Assume that the theorem is true when $|X| = n - 1$. Now we prove the theorem for $|X| = n$. Let (a_{ij}) be an integer matrix and $h_1, \ldots, h_p$ be integers satisfying the hypotheses of the theorem.

Let $\epsilon_{ij} = \frac{h_i}{n} a_{ij}$, and apply Lemma 4.2.3 to obtain a new matrix (e_{ij}). Then this new matrix satisfies the following properties:

(1) If $h_i = 0$ then $e_{ij} = 0$ since $\epsilon_{ij} \approx e_{ij}$.

(2) If $h_i = n$ then $a_{ij} - e_{ij} = 0$ since $\epsilon_{ij} \approx e_{ij}$.

(3) $e_{ij} \geq 0$ since $0 \leq h_i$, and $a_{ij} \geq 0$, and $\epsilon_{ij} \approx e_{ij}$.

(4) $a_{ij} - e_{ij} \geq 0$ since $0 \leq h_i \leq n$, and $a_{ij} \geq 0$, and $\epsilon_{ij} \approx e_{ij}$.

(5) $\sum_j e_{ij} = \binom{n-1}{h_i-1}$ since $\sum_j a_{ij} = \binom{n}{h_i}$ and $\sum_j \epsilon_{ij} \approx \sum_j e_{ij}$, and $\sum_j \epsilon_{ij} = \sum_j \frac{h_i}{n} a_{ij} = \frac{h_i}{n}\binom{n}{h_i}$.

(6) $\sum_j (a_{ij} - e_{ij}) = \binom{n-1}{h_i}$ since $\sum_j (a_{ij} - e_{ij}) = \binom{n}{h_i} - \binom{n-1}{h_i-1} = \binom{n-1}{h_i}$.

(7) $\sum_i e_{ij} \approx \frac{\sum_i h_i a_{ij}}{n}$ since $\sum_i e_{ij} \approx \sum_i \epsilon_{ij}$.

Now, we will make a new collection of data. Let $x \in X$, and let

$$\begin{aligned}
&X^* = X \backslash x && \\
&p^* = 2p && s^* = s \\
&h_i^* = h_i && h_{i+p}^* = h_i - 1 \\
&a_{ij}^* = a_{ij} - e_{ij} && a_{(i+p)j}^* = e_{ij}
\end{aligned}$$

Observe that the new data satisfy the the hypotheses of Thereom 4.0.5:

(1*) $0 \leq h_i^* \leq n-1$ for $i \in [p^*]$. If this is not true, then from (1) and (2), we have that $a_{ij}^* = 0$ for each j. Then no subset belongs to that row, so it can be ignored.

(2*) $a_{ij}^* \geq 0$ follows from (3) and (4).

(3*) $\sum_j a_{ij}^* = \binom{n-1}{h_i^*}$ follows from (5) and (6).

Then by the induction hypothesis, there exist sets F_{ij}^ν and G_{ij}^μ subsets of X^*, where $i \in [p]$, $j \in [s]$, $\nu \in [a_{ij} - e_{ij}]$ and $\mu \in [e_i j]$ so that

(a*) $|F_{ij}^\nu| = h_i$ and $|G_{ij}^\mu| = h_i - 1$.

(b*) If $j_1 \neq j_2$ or $\nu_1 \neq \nu_2$, then $F_{ij_1}^{\nu_1} \neq F_{ij_2}^{\nu_2}$, and if $j_1 \neq j_2$ or $\mu_1 \neq \mu_2$, then $G_{ij_1}^{\mu_1} \neq G_{ij_2}^{\mu_2}$.

(c*) The sets $F_{ij}^{\nu}, G_{ij}^{\mu}$ where j is fixed form an almost regular hyper graph with the set X^*.

Let us define E_{ij}^{ν} by

$$\begin{aligned} E_{ij}^{\nu} &= F_{ij}^{\nu} && \text{if } \nu \in [a_{ij} - e_{ij}] \\ E_{ij}^{\nu} &= G_{ij}^{\nu-(a_{ij}-e_{ij})} \cup \{x\} && \text{if } \nu \in \{a_{ij} - e_i j + 1, \ldots, a_{ij}\}. \end{aligned}$$

Then $E_{ij} \subset X$ satisfy:

(a) $|E_{ij}^{\nu}| = h_i$ (follows from (a*))

(b) If $j_1 \neq j_2$ or $\nu_1 \neq \nu_2$, then $E_{ij_1}^{\nu_1} \neq E_{ij_2}^{\nu_2}$ (follows from (b*))

(c) At any fixed j, the E_{ij}^{ν} for an almost regular hypergraph on X: For any fixed j, let $\mathcal{H}_j$ denote the hypergraph formed by the E_{ij}^{ν}. From the definition of the E_{ij}^{ν}, it is clear that

$$v_{\mathcal{H}_j}(x) = \sum_i e_{ij}.$$

Then from (a),

$$\frac{\sum_{i,\nu} |E_{ij}^{\nu}|}{n} = \frac{\sum_i h_i a_{ij}}{n}.$$

Combining this with (7), we find that

$$v_{\mathcal{H}_j} \approx \frac{\sum_{i,\nu} |E_{ij}^{\nu}|}{n}.$$

so, (c*) implies that $\mathcal{H}_j$ satisfies Lemma 4.2.2, so $\mathcal{H}_j$ is almost regular.

This completes the proof of Theorem 4.0.5.

Chapter 5

The Wreath Conjecture and Recent Progress

There is a conjecture of Baranyai and Katona given in [24] which gives an extension of the Baranyai theorem. Later, we will show that this conjecture implies the Erdős-Ko-Rado Theorem. For now, I will discuss the conjecture, and recent progress towards its proof.

Definition 5.0.4. Let X be a set of n elements, and give the elements an ordering: $X = \{x_1, \ldots, x_n\}$. Let $w = n/\gcd(n,k)$, and given $i \in [w]$, define

$$f_i = \{x_{(i-1)k+1}, \ldots, x_{(i-1)k+k}\},$$

where the indices are considered mod n. The family $\{f_1, \ldots, f_w\}$ is called a *wreath*. We remark that the wreath is dependent on the ordering of the elements in X.

Conjecture 5.0.5 (Baranyai & Katona, [24]). *There is a $\frac{k}{(n,k)}$-factorization of K_n^k such that each factor is a wreath.*

We will refer to this conjecture as the wreath conjecture, and such a factorization is called a *wreath decomposition*.

Example Any factorization of K_n^k is a wreath decomposition, because when k divides n, the wreaths will be 1-factors of K_n^k. Therefore, when k divides n, the Baranyai theorem is exactly the wreath conjecture. If $n = 7$ and $k = 2$, we have a wreath decomposition of K_7^2 in Figure 5.1.

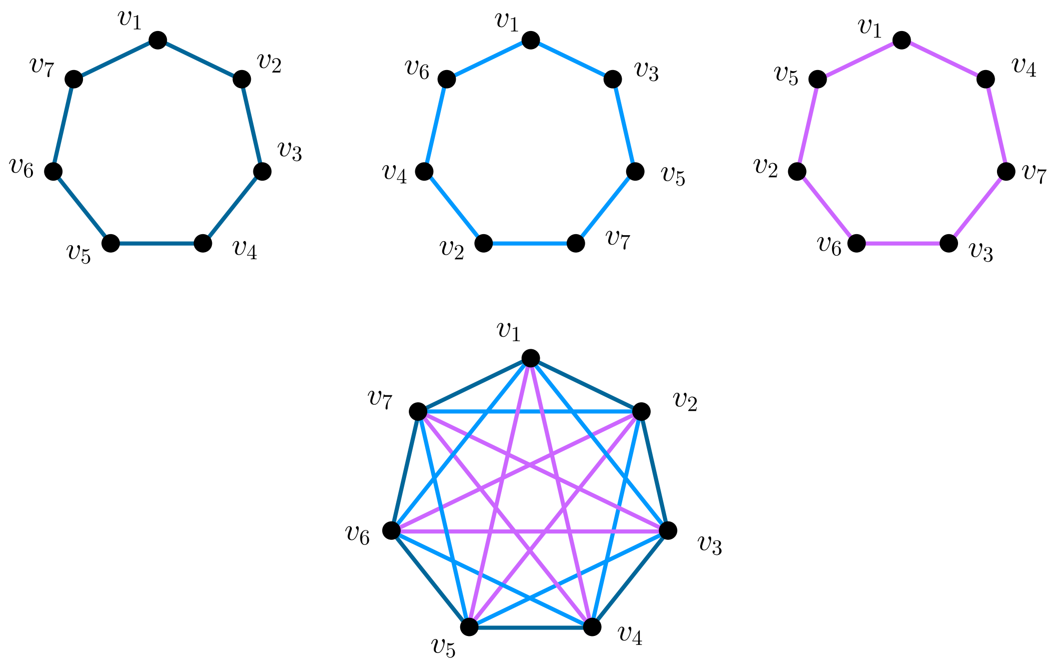


Figure 5.1: A wreath decomposition of K_7^2

5.1 Connected Baranyai Theorem

In [1] , Bahmanian gives a connected version of the Baranyai theorem by showing that a factorization for K_n^k exists. First, we set up the terminology to state Bahmanian's result.

Definition 5.1.1. An $(r_1, \ldots, r_s)$-factorization of of a hypergraph is a partition of the edge set into $\{F_1 \ldots, F_s\}$ such that each F_i is an r_i- factor.

Here, we must slightly modify the definition of a hypergraph to allow multiple copies of the same edge. Denote by $\lambda\mathcal{H}$ the hypergraph in which every edge $e \in \mathcal{H}$ is replaced with λ copies of e.

Theorem 5.1.2 (Bahmanian [1])**.** *The graph λK_n^k is $(r_1, \ldots, r_s)$-factorizable if and only if s divides $r_i n$ for $1 \leq i \leq s$, and*

$$\sum_{i=1}^{s} r_i = \lambda \binom{n-1}{k-1}.$$

Moreover, we can guarantee that if $r_i \geq 2$, then the r_i-factor is connected.

The following result strengthens this, and gives a conjecture which expands the wreath conjecture.

5.2 Hamilton-Berge Cycles

In [27] , Kühn and Osthus give the following related results and provide a conjecture which would extend the wreath conjecture.

Definition 5.2.1. A *Berge Cycle* is a sequence $v_1, e_1, v_2, \ldots, v_n, e_n$ of vertices v_i and edges e_i so that $v_i, v_{i+1} \in e_i$. Such a cycle is also called a *Hamilton* cycle of $\mathcal{H}$ if $v_1, \ldots, v_n$ is the vertex set of $\mathcal{H}$.

Theorem 5.2.2 (Kühn & Osthus, [27])**.** *Suppose that $4 \leq k < n$, and $n \geq 30$, and that n divides $\binom{n}{k}$. Then K_n^k decomposes into Hamilton-Berge cycles.*

Theorem 5.2.3 quickly implies Theorem 5.2.2.

Theorem 5.2.3 (Kühn & Osthus, [27])**.** *Let $k, n \in \mathbb{N}$, and let $3 \leq k < n$.*

1. *Suppose $k \geq 5$ and $n \geq 20$ or $k = 4$ and $n \geq 30$. Let M be any set of less than n edges of K_n^k such that n divides $|K_n^k \backslash M|$. Then the graph $|K_n^k \backslash M|$ decomposes into Hamilton-Berge cycles.*

2. *Suppose $k = 3$, and $n \geq 100$. If $\binom{n}{3}$ is not divisible by n, let M be any perfect matching in K_n^k. Otherwise, let $M = \emptyset$. Then $|K_n^k \backslash M|$ decomposes into Hamilton-Berge cycles.*

Here, we give a partial proof of Theorem 5.2.3. The main structure of the proof will be clear, but we make an assumption at one point that is proved using several cases in the paper. First, we need a result from Tillson in [33] on Hamilton decompositions of complete digraphs. The *complete digraph* DK_n on n vertices is a 2-uniform complete hypergraph where the edges are directed. So, the edges are ordered pairs (x, y) for every $x, y \in X$. A Hamilton decomposition in this context is the same as a Hamilton-Berge cycle for hypergraphs, except the cycle must also be directed.

Theorem 5.2.4 (Tillson, [33])**.** *The complete digraph DK_n has a Hamilton decomposition if and only if $n \neq 4, 6$.*

We also need some knowledge of bipartite graphs. A *bipartite graph* is a 2-uniform hypergraph such that the vertices can be partitioned into two *partite classes*, where no edge is a subset of one of the partite classes. A *perfect matching* is a collection of edges in the bipartite graph such that each vertex is contained in exactly one edge. A perfect matching gives a one-to-one correspondence between vertices in one partite class to vertices in the other class. If $A \subset X$, let $N(A)$ be the set of neighbors of A. Then Hall's condition states that a bipartite graph G with partite classes U and V has a perfect matching if and only if $|N(A)| \geq |A|$ for all subsets A of U.

Now we are ready to outline the proof of Theorem 5.2.3.

Proof of Theorem 5.2.3. For our purposes, the proof follows in the same way for parts 1 and 2 of the theorem. We will construct a bipartite graph, where one partite class consists of vertices corresponding to edges of K_n^k and the other partite class consists of vertices corresponding to edges from many copies of DK_n. Then, we will assume (this is what we are leaving out of the full proof) that there is a perfect matching in this bipartite graph. Then, we will apply Tillson's result to show that the existence of Hamilton cycles in one partite class gives us the Hamilton-Berge cycles in the other partite class, via the perfect matching.

To that end, let M be any set consisting of less than n edges of K_n^k, such that n divides $|K_n^k \backslash M|$. Let

$$l = \left\lfloor \frac{\binom{n}{k} - |M|}{n(n-1)} \right\rfloor \qquad \text{and} \qquad m = \frac{\binom{n}{k} - |M| - ln(n-1)}{n}.$$

Then l will be the number of copies of DK_n will cover almost all of $K_n^k \backslash M$ (since we wish our edges in the copies of DK_n to have a one-to-one correspondence with edges in K_n^k). We have that $m \cdot n$ will be the number of "leftover" edges in K_n^k, and so m will be the number of extra cycles in DK_n we need to add to one partite class to complete the perfect matching. Then we have that $m < n-1$ and $m \in \mathbb{N}$, which is necessary since m corresponds to the number of extra cycles we need from DK_n, and there are only $n-1$ of these cycles to start with. Define the following bipartite graph D with vertex classes A_* and B, where each class has size $\binom{n}{k} - |M|$ as follows. Let $A = K_n^k$ and let $A_* = A \backslash M$. This is one vertex class. For the other, let $D_1, \ldots, D_l$ be copies of the complete digraph DK_n on n vertices. Apply Theorem 5.2.4 to find an extra m edge-disjoint Hamilton cycles $H_1, \ldots, H_m$ in DK_n. We view $D_1, \ldots, D_l, H_1, \ldots, H_m$ as being pairwise disjoint, and let B denote the union of these sets. Then

$$|B| = l(n(n-1)) + m \cdot n = \binom{n}{k} - |M| = |A_*|.$$

Then let our bipartite graph G have an edge between $z \in A_*$ and $(x, y) \in B$ if and only if $\{x, y\} \subset z$. So, we think of elements of B as being 2-shadows of edges in K_n^k (the part of the proof which we are leaving out relies on the Kruskal-Katona theorem).

Now, assume that G has a perfect matching F. This is proved in [27] using Hall's condition and applying the Kruskal-Katona theorem. Now, for each $i \in \{1, \ldots, l\}$ apply Theorem 5.2.4 to obtain a Hamilton Decomposition

$$H_i^1, \ldots, H_i^{n-1}$$

of D_i. For each $i \in \{1, \ldots l\}$ and each $j \in \{1, \ldots, n-1\}$, let $A_i^j \subset A$ be the neighborhood of H_i^j in F, our perfect matching. Then each A_i^j is the edge set of a Hamilton-Berge cycle in $K_n^k \backslash M$, because the cycle H_i^j in DK_n will induce a cycle in K_n^k, since each directed edge $\{x, y\}$ is contained in precisely one edge in K_n^k via the perfect matching. Similarly, for each $i \in \{1, \ldots, m\}$, the neighborhood A_i' of H_i in F is also the edge set of a Hamilton-Berge cycle. The A_i^j and A_i' are all pairwise disjoint because of the perfect matching, so this gives a decomposition of $K_n^k \backslash M$ into Hamilton-Berge cycles. □

Now we will discus a conjecture of Kühn & Osthus which extends the wreath conjecture.

Definition 5.2.5. A k-uniform hypergraph C is an *l-cycle* if there exists an ordering of the vertices such that every edge consists of k consecutive vertices and any pair of consecutive edges intersects at l vertices.

Conjecture 5.2.6 (Kühn & Osthus, [27])**.** *For all $k, l \in \mathbb{N}$ with $l < k$, there exists n_0 such that for all $n \geq n_0$, if $k-l$ divides n and $\frac{n}{k-l}$ divides $\binom{n}{k}$, then K_n^k decomposes into Hamilton l-cycles.*

In the case where $l = k - (n, k)$, this conjecture is the wreath conjecture.

5.3 Decompositions of Complete Uniform Graphs into $(k-1)$-Cycles

In [2], Bailey and Stevens define a Hamiltonian decomposition and demonstrate some methods for computing them.

A clear necessary condition for the decomposition of K_n^k into $(k-1)$-cycles is that n divides $\binom{n}{k}$. Bailey and Stevens make the following conjecture, which is handled by the conjecture of Kühn & Osthus.

Conjecture 5.3.1 (Bailey & Stevens)**.** *There is a decomposition of K_n^k into $(k-1)$-cycles if and only if n divides $\binom{n}{k}$.*

In the case when n and k are relatively prime, this coincides with the wreath conjecture.

Their original method for finding decompositions of K_n^k into $(k-1)$-cycles is as follows. They made a graph $\Gamma_{n,k}$, where the vertex set was the set of all possible $(k-1)$-cycles, and two vertices shared an edge if the corresponding decompositions were disjoint (meaning, the decompositions did not share any edges). Then a clique in $\Gamma_{n,k}$ is a set of mutually disjoint cycles, so the problem is solved if we can find a clique of size $\frac{1}{n}\binom{n}{k}$. Using this, they were able to make decompositions for $(n, k) = (7, 3), (8, 3), (9, 4)$ on a regular computer. At the time they wrote the paper, a high-performance computing facility was working towards a decomposition of $(10, 4)$.

Bailey and Stevens give another method which yields solutions very quickly for some n and $k = 3$.

5.4 NP-Completeness

The following result from [12] suggests that the problem of decomposing K_n^k into wreaths may be NP-complete.

Theorem 5.4.1. *The problem of whether a hypergraph $\mathcal{H}$ can be decomposed into l connected spanning subhypergraphs is NP-complete for every integer $l \geq 2$.*

Proof. First, assume that $l = 2$. The authors note that the problem of two-coloring the vertices of a hypergraph such that there are no monochromatic edges is NP-complete. By taking the dual of the hypergraph, this implies that two-coloring the edges of a hypergraph so that every vertex is contained in edges of two colors is also NP-complete (the red and the blue edges both cover X). We claim that this problem would be polynomially solvable if there were a polynomial algorithm to decide whether a hypergraph can be decomposed into two connected spanning subhypergraphs. Let $\mathcal{H}$ be a hypergraph on X. Let $x \notin X$, and let $X' = X \cup \{x\}$. Let $\mathcal{H}' = \{e \cup \{x\} \mid e \in \mathcal{H}\}$. If $\mathcal{F}'$ is a subhypergraph of $\mathcal{H}'$, then $\mathcal{F}'$ is connected and spans X' if and only if corresponding subhypergraph $\mathcal{F}$ of $\mathcal{H}$ covers the elements of X. Therefore, $\mathcal{H}'$ can be decomposed into two connected spanning subhypergraphs if and only if $\mathcal{H}$ can be decomposed into two spanning subhypergraphs.

Now, suppose $l \geq 3$. Allow a hypergraph to potentially contain multiple copies of the same edge, and let $\mathcal{H}$ be a hypergraph, and let $\mathcal{H}^+$ denote the hypergraph $\mathcal{H} \cup \{X\} \cup \cdots \cup \{X\}$, where there are $l-2$ copies of the edge X. Then $\mathcal{H}^+$ can be decomposed into k connected spanning subhypergraphs if and only if $\mathcal{H}$ can be decomposed into two connected spanning subhypergraphs. $\square$

This result suggests that wreath decomposition may also be NP-complete. Let $\mathcal{H}$ be a hypergraph on X. Then Exact Cover is the problem of finding a factor of $\mathcal{H}$. It is known that this problem is NP-complete. We will use that fact to prove the following proposition.

Proposition 5.4.2. *The problem of factorizing a hypergraph $\mathcal{H}$ into uniform factors is NP-complete.*

Proof. Given a hypergraph $\mathcal{H}$ on a set E, define a new hypergraph $\mathcal{H}'$. If $\mathcal{H}$ has no exact cover, let $\mathcal{H}' = \{\}$ on a nonempty set X. Otherwise, let L be an exact cover of $\mathcal{H}$. Let $l = \mathrm{lcm}(\{|e| : e \in L\})$. Let $X = \{x_1, \ldots, x_l\}$. For e in L, divide X into sets of size $|e|$, and add these sets to $\mathcal{H}'$. Do this for each $e \in L$. Then each e corresponds to a factor of $\mathcal{H}'$, and L is the factorization of $\mathcal{H}'$. Therefore, $\mathcal{H}$ has an exact cover if and only if $\mathcal{H}'$ has a factorization where each factor is uniform. $\square$

5.5 Finding Examples for the Wreath Conjecture

I wrote an algorithm which randomly generates $\frac{(n,k)}{n}\binom{n}{k}$ cyclic permutations of $\{1, \ldots, n\}$, and then checks to see if the corresponding wreaths form a decomposition (since any permutation of the vertices will give a wreath). See Appendix A.1 for the *Mathematica* code. This algorithm was able to check about 1,000 decompositions per second. Fairly quickly, it gives solutions for the small examples: $(n, k) \in \{(4, 2), (5, 2), (6, 2), (6, 3)\}$. We do not worry about $k > n/2$, since these decompositions can be obtained by taking the complements of edges in the decomposition of $(n, n-k)$. The pair $(7, 3)$ takes longer to find a decomposition for (it ran about 2.5 million examples, taking around 45 minutes). I ran my computer overnight on $(8, 3)$, and no solutions were found. So far, I have found 20 decompositions of $(7, 3)$, and 18 of them were distinct (i.e., two were repeats of the others).

Here is one of the decompositions of K_7^3, where each row is a wreath:

$$
\begin{array}{ccccccc}
\{1,2,3\} & \{2,3,4\} & \{3,4,5\} & \{4,5,6\} & \{5,6,7\} & \{1,6,7\} & \{1,2,7\} \\
\{3,4,7\} & \{3,4,6\} & \{2,4,6\} & \{1,2,6\} & \{1,2,5\} & \{1,5,7\} & \{3,5,7\} \\
\{1,3,5\} & \{1,3,4\} & \{1,4,6\} & \{4,6,7\} & \{2,6,7\} & \{2,5,7\} & \{2,3,5\} \\
\{4,5,7\} & \{2,4,5\} & \{2,5,6\} & \{2,3,6\} & \{1,3,6\} & \{1,3,7\} & \{1,4,7\} \\
\{2,3,7\} & \{3,6,7\} & \{3,5,6\} & \{1,5,6\} & \{1,4,5\} & \{1,2,4\} & \{2,4,7\}
\end{array}
$$

5.6 The Johnson Graph

The problem of decomposing complete uniform hypergraphs into Hamilton-Berge cycles was first solved in the 3-uniform case in the papers by J.C. Bermond [4] and Verrall [34]. Bermond solved the case when $n \equiv 2 \mod 3$ and $n \equiv 4 \mod 6$, while Verrall completed the result by proving it when $n \equiv 1 \mod 6$ or when $n \equiv 0 \mod 3$ and the graph to be decomposed was $K_n^3 - I$, where I is a 1-factor of K_n^3. In both cases, the authors used something called a *choice design* which gives a distinguished element for every triple, and then used theorems for graphs to obtain cycle decompositions, and then added to each edge in these decompositions the vertices selected by the choice design. They proved that this process gave a Hamilton-Berge decomposition of K_n^3.

In a similar fashion, I decided to start with the case $k = 3$ and try to translate the question of a wreath decomposition of K_n^3 into a question about graph theory, and then hope that there were known results in graph theory that would give the desired decompositions. This led me to the following.

Definition 5.6.1. Let $1 \leq k < n/2$. The *Johnson Graph* $J(n,k)$ is the graph whose vertex set is the edges of K_n^k, and any two vertices v, w form an edge whenever $|v \cap w| = k - 1$.

Definition 5.6.2. Let $G = (V, E)$ be a graph, and let $W \subset V$. Then the *induced subgraph at* W is the graph (W, E_W), where E_W is the set of all $e \in E$ such that $e \subset W$.

Then, if n, k are relatively prime, K_n^k has a wreath decomposition if and only if there exists a partition $P = \{W_1, \ldots, W_{\binom{n}{k}/n}\}$ of the vertices of $J(n,k)$ into n-sets such that the induced subgraph at each W_i is a cycle. Of course, the W_i would be precisely the wreaths in the wreath decomposition. For example, in the case $n = 5, k = 3$, we have the cycles in $J(5,3)$ given by Figure 5.2, and the cycles in $J(7,3)$ corresponding to the wreaths from the previous section in Figure 5.3. See Appendix A.4 for the code used to make these figures.

Then the question is: what do we know about the Johnson graph?

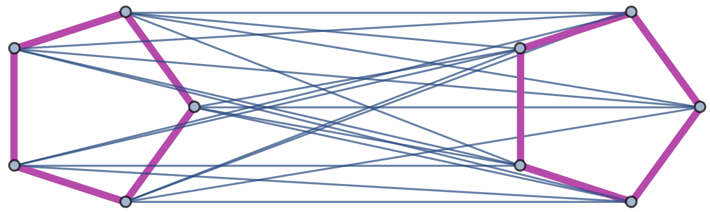


Figure 5.2: The cycle partition for $J(5,3)$.

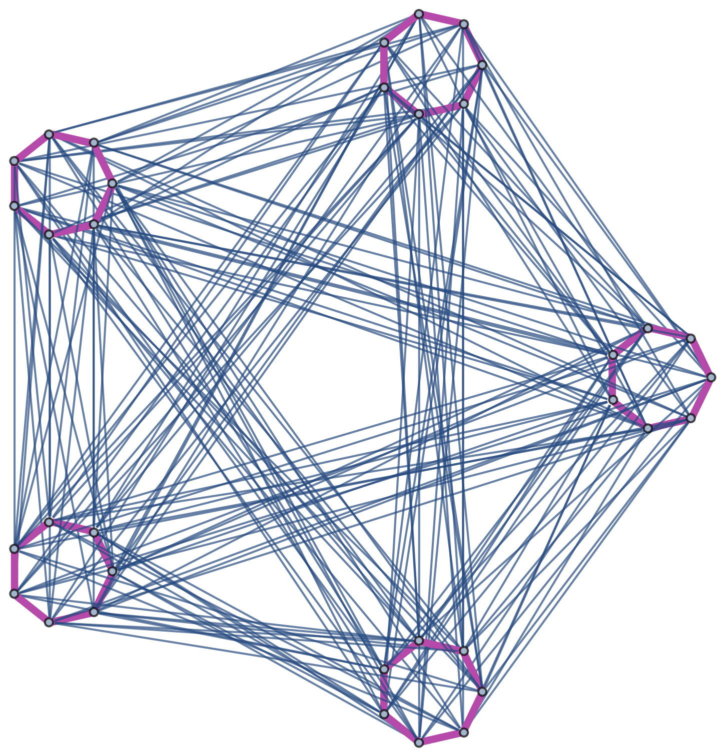


Figure 5.3: The cycle partition for $J(7,3)$.

Figure 5.4 displays the Johnson graphs for $k \in \{1,2,3\}$, $n \in \{4,5,6,7\}$. For some special values of n, k, the Johnson graphs are familiar. We have that $J(n,1)$ is the complete graph, $J(n,2)$ is called the *triangular graph*, and $J(n,3)$ is called the *tetrahedral graph*. The complement of $J(5,2)$ is the *Petersen graph*. In general, $J(n,2)$ is the complement of the *Kneser graph*, $K(n,2)$, which has the same vertex set as $J(n,2)$ but the vertices are connected when the corresponding sets are disjoint. The Johnson graphs are *distance transitive*, which means that if x, y are vertices at distance d, and w, v are vertices at distance d, then there is an automorphism of the graph which takes x to w and y to v. Any pair of vertices in the Johnson graph are the endpoints of a Hamiltonian path in the graph.

The following question was studied by Naimi and Shaw in [31]. When is a graph

G an induced subgraph of a Johnson graph? They proved that all trees, cycles, and complete graphs are induced subgraphs of some Johnson graph.

Ramras and Donovan in [32] studied the automorphism group of the Johnson graph. When $n \neq 2k$, the automorphism group is S_n, where S_n acts on the vertices by permuting the elements in the vertices. When $n = 2k$, they conjecture that the automorphism group is $S_n \times \langle T \rangle$, where T takes an edge to its complement. Ganesan proves that this is true in [18].

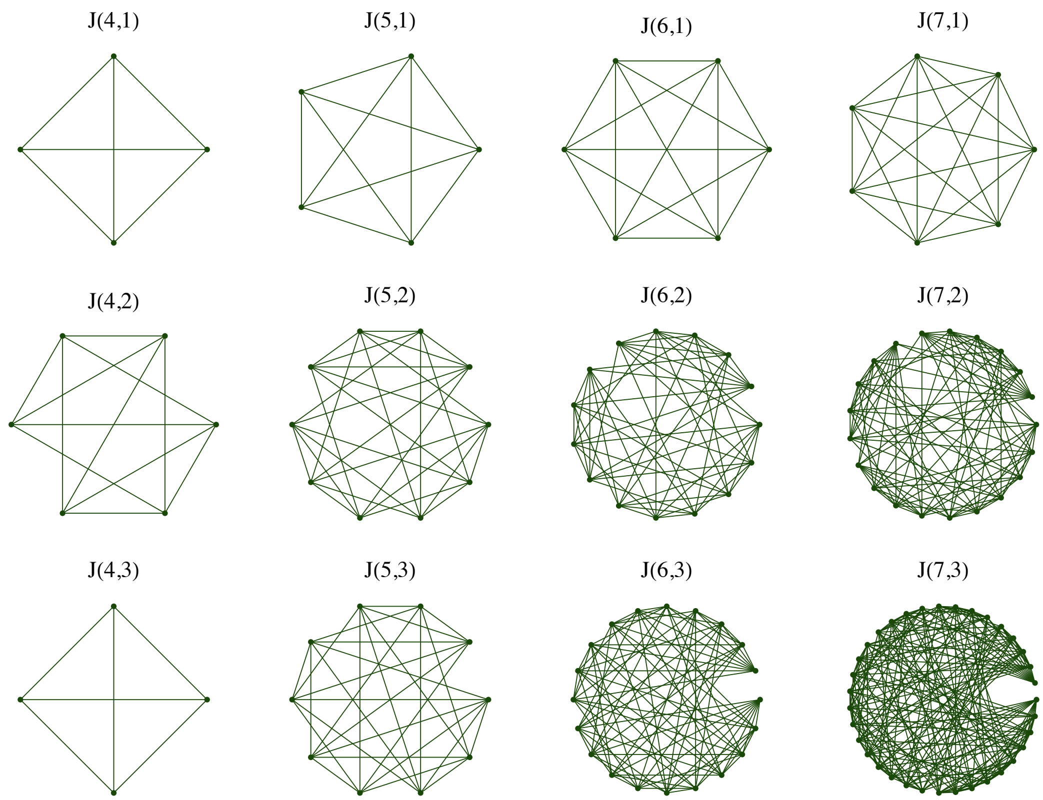


Figure 5.4: $J(n,k)$ for $k \in \{1,2,3\}$, $n \in \{4,5,6,7\}$.

Chapter 6

Wreath Conjecture Implies Erdős-Ko-Rado Theorem

In this thesis, we had hoped to expand the proof that the Baranyai theorem implies the Erdős-Ko-Rado theorem in the case when k divides n (see Section 1.3) to a full proof of the Erdős-Ko-Rado theorem. However, we found that the Baranyai theorem alone was not strong enough to prove the Erdős-Ko-Rado theorem in the way that we had hoped to use it. First, we will demonstrate why the Baranyai theorem was not strong enough to prove the Erdős-Ko-Rado theorem. Then, we will show that the wreath conjecture implies the Erdős-Ko-Rado theorem. The contents of this chapter are original to this thesis.

6.1 Weakness of the Baranyai Theorem

We had hoped to use the following version of the Baranyai theorem to prove the Erdős-Ko-Rado theorem. Let $e_1, \dots, e_s$ be non-negative integers such that $\sum e_i = \binom{n}{k}$. Then $K_n^k = \mathcal{H}$ can be partitioned into s sets $\mathcal{H}_1, \dots, \mathcal{H}_s$ such that $|\mathcal{H}_i| = e_i$ and every hypergraph $\mathcal{H}_i$ is almost-regular. Then a proof of the Erdős-Ko-Rado Theorem from this theorem would go as follows. Given n and k we would specify the e_i. Then for each $\mathcal{H}_i$, we would put a bound on the size of an intersecting hypergraph which sits inside of $\mathcal{H}_i$. Call this number $s(e_i)$. Then we would show that $\sum(s(e_i)) \leq \binom{n-1}{k-1}$. Note that this is the way the proof works when k divides n: all e_i are n/k, and then $s(e_i) = 1$ (see Section 1.3).

Now, I will demonstrate that in the case of $n = 7$, $k = 3$, there is no suitable selection of e_i. First, note that a selection of e_i corresponds to a partition of $35 = \binom{7}{3}$. What are the values of $s(n)$? First, we observe that $s(n)$ is increasing, and $s(n) \leq n$, and $s(n) \leq 15$. By drawing small examples, we determine that

$$s(1) = 1, \quad s(2) = 1, \quad s(3) = 2, \quad s(4) \geq 3, \quad s(5) \geq 4, \quad s(6) \geq 6, \quad s(7) = 7$$

where $s(7) = 7$ because of the Fano plane, which is a 3-uniform, intersecting hypergraph with 7 edges (see Figure 6.1). We do not need to determine the exact values of $s(n)$, as it turns out that these estimates will be enough. Then by the increasing

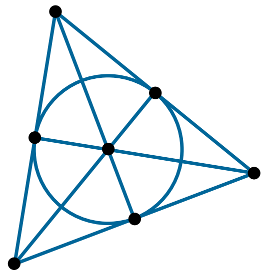


Figure 6.1: The Fano Plane

property, we know $s(i) \geq 7$ for $7 \leq i \leq 13$. Then $s(14) = 14$, because it is possible to layer two disjoint Fano planes on top of one another. Then $s(i) = 14$ for $14 \leq i \leq 32$. These values are exact, since we know that by the Erdős-Ko-Rado Theorem, 15 is the maximal size for an intersecting graph, and in this case, one vertex will have degree 15, so the others have degree at least 14 in order for the factor to be almost regular. Then $s(33) = s(34) = 15$.

Now, for all partitions $\lambda = (e_1, \ldots, e_s)$ of 35, we compute $\sum s(e_i)$ (using the lower bounds for s determined above). By running this on the computer (see Appendix A.2), we find that none of these values is less than or equal to 15, and so there do not exist e_i which we can use to prove the theorem.

This shows that we need to know something about the structure of the factors in order to improve the bounds on the number of edges which can be taken from each factor. We note that none of the extensions of the Baranyai theorem discussed in the previous chapter will suffice. The strongest of these results states that K_n^k can be decomposed into Hamilton-Berge cycles, but in the example above, this would leave the values of $s(n)$ the same.

6.2 Wreath Conjecture Implies Erdős-Ko-Rado Theorem

Since the wreath conjecture gives much more information about the structure of the factors, we can say more about the maximal size of an intersecting hypergraph inside of each of the factors.

Proposition 6.2.1. *The Wreath Conjecture implies the Erdős-Ko-Rado Theorem.*

Proof. Suppose the Wreath Conjecture is true, meaning the edges K_n^k can be partitioned into disjoint wreaths:

$$K_n^k = W_1 \cup \cdots \cup W_l.$$

Let $n > 2k$. We observe that each wreath is k-uniform, $\frac{k}{(n,k)}$-regular, has $\frac{n}{(n,k)}$ edges, and is on n vertices. Additionally, there are $\frac{(n,k)}{n}\binom{n}{k}$ wreaths in the decomposition.

Let $\mathcal{H}$ be a k-uniform, intersecting hypergraph on n vertices. We view $\mathcal{H}$ as a subgraph of K_n^k.

Claim 6.2.2. *An intersecting subhypergraph of a wreath has at most $\frac{k}{(n,k)}$ edges.*

Proof of claim. Let W be a wreath, and let $\mathcal{F} \subset W$ be an intersecting hypergraph. For $w_i \in W$, define

$$I(w_i) = \{w_j \in W \mid w_i \cap w_j \neq \emptyset\}.$$

Then for any $f_i \in \mathcal{F}$, we have that $\mathcal{F} \subset I(f_i)$. Then, $\mathcal{F} \subset \cap_i I(f_i)$. We observe that (using the fact that $n > 2k$)

$$|I(w_i)| = 2\frac{k}{(k,n)} - 1,$$

and that

$$| \cap_{i=1}^{l} I(w_i)| \leq 2\frac{k}{(k,n)} - l.$$

This is because after w_1, for each w_i we add, we must remove from $I(w_1)$ the unique edge in $I(w_1)$ which is the complement of w_i in w_1. Then,

$$|\mathcal{F}| \leq | \cap_i I(f_i)| \leq 2\frac{k}{(n,k)} - |\mathcal{F}|,$$

which implies that $|\mathcal{F}| \leq \frac{k}{(n,k)}$. □

Then by Claim 6.2.2, for each wreath W_i, we have that $|W_i \cap \mathcal{H}| \leq \frac{k}{(n,k)}$, so that

$$|\mathcal{H}| = | \cup_i (W_i \cap \mathcal{H})| \leq \left(\frac{k}{(n,k)}\right)\left(\frac{(k,n)}{n}\binom{n}{k}\right) = \binom{n-1}{k-1}.$$

□

Appendix A

Mathematica Code

Here, I will present the *Mathematica* code I wrote during this thesis. This includes algorithms to find wreath decompositions, my computations demonstrating the weaknesses of the Baranyai theorem, the shifting operation, and the code which generates the figures related to the Johnson graphs.

A.1 Finding Wreath Decompositions

MakeWreath takes in n, k, and an ordering of n elements, and returns the k-uniform wreath on n elements under that ordering (with the first edge as the first k elements listed in the ordering)

```
MakeWreath[n_, k_, pi_] :=
 Table[Sort[
   Table[pi[[Mod[i + j GCD[n, k], n] + 1]], {i, 0, k - 1}]], {j, 0,
   n/GCD[n, k] - 1}]
```

CompleteGraph takes in n, k and makes the complete k-uniform graph on n vertices.

```
Completenk[n_, k_] := Subsets[Table[i, {i, 1, n}], {k}]
```

SetMake will take a list and make *Mathematica* treat it like a set: order elements and remove duplicates.

```
SetMake[L_] := Sort[DeleteDuplicates[L]]
```

TestWreath will test if a wreath is compatible with the decomposition already formed, where K is the set of wreaths.

```
TestWreath[K_, W_] := Length[Intersection[Flatten[K, 1], W]] == 0
```

FindDecomp takes in n, k and makes a wreath decomposition by testing each permutation of n elements to decide if the corresponding wreath fits in with the collection already formed. Then, it either adds in the new wreath or throws it out correspondingly, and tests the next permutation. This algorithm will not necessarily give a wreath decomposition. In many cases, it will only give a partial decomposition.

```
FindDecomp[n_, k_] :=
 Module[{Pinitial, P, Wreaths, g, GoodPerms, W, i, K},
  P = Permutations[Table[i, {i, 1, n}]];
  GoodPerms = {};
  Wreaths = {};
```

```
  i = 1;
  K = Completenk[n, k];
  While[Length[Wreaths] < WreathNumber[n, k] && i < Length[P],
   W = MakeWreath[n, k, P[[i]]];
    If[TestWreath[Wreaths, W], AppendTo[GoodPerms, P[[i]]]];
    If[TestWreath[Wreaths, W], AppendTo[Wreaths, W]];
    i = i + 1;
    ];
  Return[{Wreaths, i, GoodPerms}];
  ]
```

WreathNumber takes in n, k and gives the number of wreaths needed for a wreath decomposition.

```
WreathNumber[n_, k_] := GCD[n, k]/n Binomial[n, k]
```

RandomWreathDecomp makes wreath decompositions by randomly selecting WreathNumber[n,k] permutations and tests to see if this is a wreath decomposition.

```
RandomMakeDecomp[n_, k_] := Module[{Decomp, test, perms},
  Decomp = {};
  i = 1;
  While[Length[Decomp] == 0,
   perms =
    Join[{Table[i, {i, 1, n}]},
     Table[Permute[Table[i, {i, 1, n}], RandomPermutation[n]], {j, 1,
       WreathNumber[n, k] - 1}]];
   test =
    Table[MakeWreath[n, k, perms[[j]]], {j, 1, WreathNumber[n, k]}];
   If[Length[Union[Flatten[test, 1]]] == Binomial[n, k],
    Decomp = {test, perms}
    ];
   i = i + 1;
   ];
  Return[Decomp];
  ]
```

SameDecomp tests to see if two wreath decompositions are the same. Here, a decomposition is $\{D, P\}$, where D is the edges of the wreath decomposition and P is the set of permutations that generated the wreaths. It returns true if they are the same, false if they are different.

```
SameDecomp[D1_, D2_] := Module[{De1, De2, T, c},
  De1 = D1[[1]];
  De2 = D2[[1]];
  T = Table[
    Length[Union[De1[[i]], De2[[j]]]], {i, 1, Length[De1]}, {j, 1,
     Length[De1]}];
  c = Count[Flatten[T], Length[De1[[1]]]];
  If[c == Length[De1], Return[True], Return[False]]
  ]
```

A.2 Weakness of Baranyai Theorem

The following demonstrates that the Baranyai Theorem cannot be used in the way we wished to prove the Erdős-Ko-Rado Theorem. The function s is as defined in Section 6.1.

```
A = IntegerPartitions[35, 35, Table[i, {i, 1, 34}]];
B = Table[If[Total[Map[s, A[[i]]]] <= 15, 1, 0], {i, 1, Length[A]}];
Count[B, 1]
```

A.3 Shifting

The function S defined below takes in two numbers i, j and a hypergraph F and performs the shifting operation on F.

```
S[{i_, j_}, F_] := Module[{N, H, HH, l},
  N = Length[F];
  HH = {};
  H = Table[Sort[F[[k]]], {k, 1, N}];
  For[l = 1, l <= N, l = l + 1,
    If[MemberQ[H[[l]], j],
      If[! MemberQ[H[[l]], i],
        If[! MemberQ[H,
            Sort[Append[Delete[H[[l]], Position[H[[l]], j]], i]]
            ],
         HH =
          AppendTo[HH,
            Sort[Append[Delete[H[[l]], Position[H[[l]], j]], i]]],
         HH = AppendTo[HH, H[[l]]]
         ],
       HH = AppendTo[HH, H[[l]]]
       ],
     HH = AppendTo[HH, H[[l]]]
     ];
   ];
  Return[HH];
  ]
```

A.4 Johnson Graphs

J takes in n, k and gives the Johnson Graph $J(n, k)$.

```
J[n_, k_] :=
 Graph[Subsets[Table[i, {i, 1, n}], {k}],
  DeleteCases[
   Flatten[Table[
     If[Length[
         Intersection[Subsets[Table[i, {i, 1, n}], {k}][[i]],
          Subsets[Table[i, {i, 1, n}], {k}][[j]]]] == k - 1,
      Subsets[Table[i, {i, 1, n}], {k}][[i]] <->
       Subsets[Table[i, {i, 1, n}], {k}][[j]], 1], {i, 1,
      Binomial[n, k]}, {j, i + 1, Binomial[n, k]}], 1], _Integer]]
```

The following set of commands help make figures of the cycle partitions of the Johnson Graphs.
VPosition takes in n, k and gives a table of vertex coordinates for the graph.

```
VPosition[n_, k_] := Module[{w, c0, r},
  w = Binomial[n, k]/n;
  r = 8 w/(2 \[Pi]);
  c0 = Table[{r Cos[2 \[Pi] j /w], r Sin[2 \[Pi] j /w]}, {j, 1, w}];
  Return[Table[
    Table[{Cos[2 \[Pi] j /n] + c0[[l, 1]],
      Sin[2 \[Pi] j /n] + c0[[l, 2]]}, {j, 1, n}], {l, 1, w}]]
  ]
```

CyclePic takes in n, k and the list of permutations that generate a wreath decomposition and returns a picture of the cycle partition of the Johnson Graph $J(n, k)$.

```
CyclePic[n_, k_, perms_] := Module[{cycles, VCoords, vpos},
  cycles = Flatten[Table[Table[
      Style[
       Sort[Table[perms[[j, Mod[c, n] + 1]], {c, i, k + i - 1}]] <->
```

```
        Sort[Table[
          perms[[j, Mod[c, n] + 1]], {c, i + 1, k + i}]], {Thickness[
         0.01], Red}]
     , {i, 1, n}], {j, 1, Length[perms]}], 1];
vpos = VPosition[n, k];
VCoords = Flatten[Table[Table[
     Sort[Table[perms[[l, Mod[c, n] + 1]], {c, i, k + i - 1}]] ->
      vpos[[l, i]]
     , {i, 1, n}], {l, 1, Length[perms]}], 1];
Return[HighlightGraph[J[n, k], cycles, VertexCoordinates -> VCoords]]
]
```

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