Robots and Geometric Theorem Proving

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Topics in Algebra Project

Robots

1.1 Introduction

The material and techniques we have learned this semester may be applied to an interesting problem in robotic motion planning. A systematic approach can be developed for understanding the possible configurations of a "robot arm" with specified segments and joints. Several simplifying assumptions about robots and their possible components are made in order to examine them geometrically. These include that the robot lives in a plane, that its arm has rigid segments, and that its joints are of specific types. We also assume that one end of the robot is held fixed, while the other end is a "hand" for which we may wish to know the possible positions. The goal is to describe the correspondence between configurations of the arm and the possible hand locations.

The motion of the robot is determined by the possible motions of the joints. We will consider two types of joints: revolute joints and prismatic joints. Revolute joints

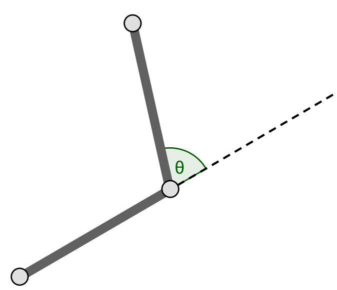


Figure 1.1: A revolute joint.

can rotate, while prismatic joints can extend in length. The position of a revolute joint occurring between segments i and $i+1$ is described by an angle $\theta \in [0, 2\pi] = S^1$ measured counterclockwise from segment i to segment $i+1$ (Figure 1.1). The position of a prismatic joint is specified by its total length, $l \in [0, l_{max}] \subseteq \mathbb{R}$ (Figure 1.2). A robot is specified by segments and joints; the segments and joints of a robot are numbered in increasing order from the fixed end of the robot to the hand (Figure 1.3).

Each of the joint settings can be set independently, so the possible settings of the joints on a robot with r revolute joints and p prismatic joints is parametrized by a

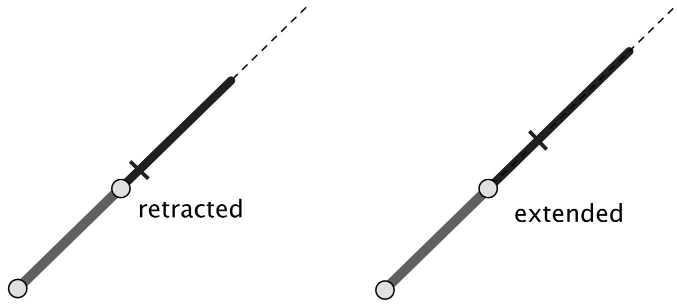


Figure 1.2: A prismatic joint in two configurations.

Cartesian product containing r copies of S^1 and p intervals corresponding each to one of the prismatic joints:

$$\mathcal{J} = S^1 \times \cdots \times S^1 \times I_1 \times \cdots I_p.$$

We call $\mathcal{J}$ the *joint space* of the robot.

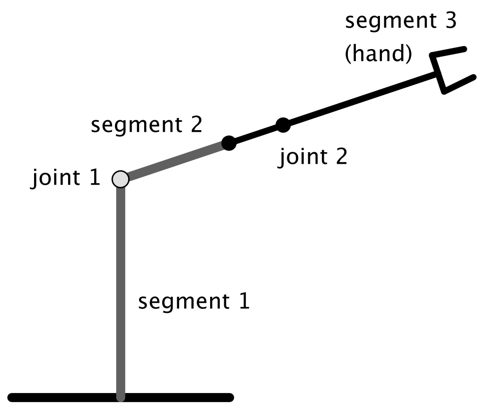


Figure 1.3: An example of a robot.

We can also describe the space of possible configurations of the hand. The hand will have both a location and an orientation. We let $U \subset \mathbb{R}^2$ be the set of possible locations of the hand, and we let $V = S^1$ be the set of possible orientations of the hand. Then we call $C = U \times V$ the *configuration space* of the hand.

Each collection of joint settings will uniquely determine a hand location and orientation. This gives rise to

$$f : \mathcal{J} \to C,$$

which describes how the joint settings yield hand configurations.

Two main problems will be considered:

- The **Forward Kinematic Problem** is to find an explicit formula for f given the joint settings and the lengths of the robot arms.
- The **Inverse Kinematic Problem** is to find, given $c \in C$, the $j \in \mathcal{J}$ such that $f(j) = c$. There may be multiple j which accomplish this, or none whatsoever!

The forward kinematic problem can be solved relatively easily, and it will be shown that f may be written as a polynomial mapping. The inverse problem is more difficult, as we will need to solve the equation $f(j) = c$. It might be that there is more than one j which works. In some cases, this is desirable. We may wish specify certain barriers for the robot to work around, in which case some of the solutions will not actually be physically realizable. To determine if it is possible to reach a specific hand configuration, then, we might determine all solutions to $f(j) = c$, and then check which of those work given the constraints of the robot's environment.

1.2 Forward Kinematic Problem

The goal for this section is to provide a standard method for solving the forward kinematic problem for a given robot. That is, we wish to find a way to generate the mapping $f : \mathcal{J} \to C$ for a robot, so that we may determine the location of the hand provided a specific configuration of the joints of the robot. We will begin by finding this function trigonometrically, and then converting this trigonometric formulation into an algebraic one.

First, we must define a coordinate system for the robot. The initial endpoint of segment 1 is always assumed to be fixed. Then, the cartesian coordinate system is used, with the origin fixed at joint 1 of the robot arm (note that joint 1 is fixed because segment 1 is fixed). We call this the (x_1, y_1) coordinate system.

Next, we will define a local coordinate system for each revolute joint as follows. Consider revolute joint i. Define the (x_{i+1}, y_{i+1}) coordinate system with (refer to Figure 1.4):

- the origin located at joint i,
- the positive x_{i+1}-axis is along the direction of segment $i + 1$,
- and the positive y_{i+1}-axis forms the regular right-handed coordinate system with the x_{i+1}-axis as described.

Note that the (x_i, y_i) coordinates of joint i are $(l_i, 0)$, where l_i is the length of segment i. Now, the first thing we need to do is relate the (x_{i+1}, y_{i+1}) coordinates to the (x_i, y_i) coordinates. Once we have done this, we can relate the position of the hand to the (x_1, y_1) coordinates by translating it through each coordinate system. To that end, let θ_i be the counterclockwise angle from the x_i axis to the x_{i+1} axis. Let q be a point in the plane with (x_{i+1}, y_{i+1}) coordinates (a_{i+1}, b_{i+1}). Let the corresponding (x_i, y_i)

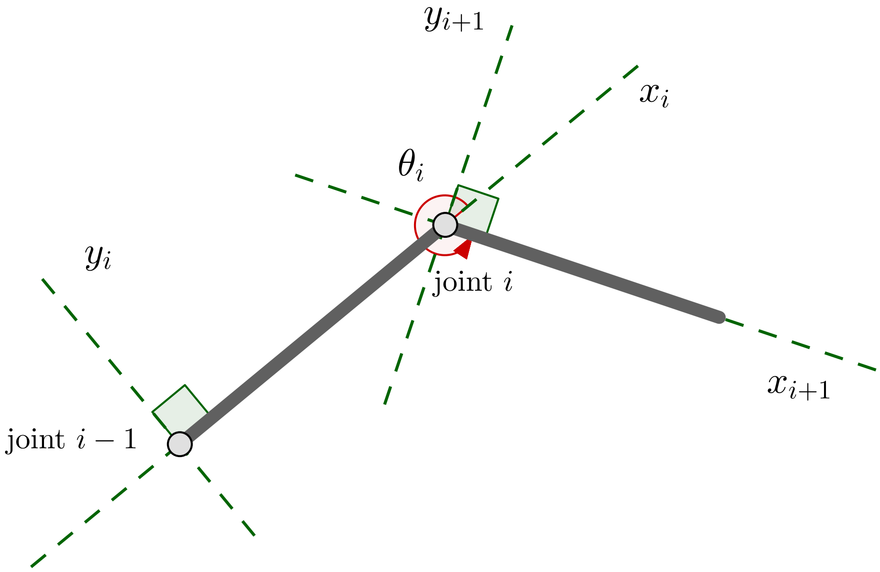


Figure 1.4: Example coordinate system.

coordinates be (a_i, b_i). Then, we must perform two operations. First, we must rotate by θ_i and then translate by $(l_i, 0)$.This can be written using matrices:

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix} + \begin{pmatrix} l_i \\ 0 \end{pmatrix}.$$

Then, this entire operation can be rewritten as one 3×3 matrix:

$$\begin{pmatrix} a_i \\ b_i \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & l_i \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ 1 \end{pmatrix}. \tag{1.1}$$

Now, we will want to convert these trigonometric expressions into algebraic ones. This can be accomplished by using the parametrization of the variety $V(x^2 + y^2 - 1)$:

$$x = \cos\theta$$
$$y = \sin\theta.$$

So, let

$$c_i = \cos\theta_i$$
$$s_i = \sin\theta_i,$$

and subject c_i and s_i to the constraint $c_i^2 + s_i^2 - 1 = 0$. Then the joint space $\mathcal{J}$ of a robot with r revolute joints is given by $V(x_1^2 + y_1^2 - 1, \ldots, x_r^2 + y_r^2 - 1)$. Geometrically, this is just r copies of the unit circle. Now, we can rewrite Equation 1.1:

$$\begin{pmatrix} a_i \\ b_i \\ 1 \end{pmatrix} = \begin{pmatrix} c_i & -s_i & l_i \\ s_i & c_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ 1 \end{pmatrix} \equiv A_i \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ 1 \end{pmatrix}. \tag{1.2}$$

Example 1.2.1. Consider a robot with 3 segments of lengths l_1, l_2, l_3 and a hand (which we think of as segment 4), with a revolute joint between each segment. Then

$$A_1 = \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} c_2 & -s_2 & l_2 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A_3 = \begin{pmatrix} c_3 & -s_3 & l_3 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where A_1 reflects the fact that the origin of the (x_2, y_2) coordinate system is at joint 1. Then the global coordinates of any point in the (x_4, y_4) coordinate system can be found by finding the coordinates of the point in preceding coordinate systems one at a time. Then,

$$\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = A_1 A_2 A_3 \begin{pmatrix} x_4 \\ y_4 \\ 1 \end{pmatrix}.$$

Since the (x_4, y_4) coordinates of the hand are $(0, 0)$, we have

$$\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = \begin{pmatrix} l_2 c_1 + l_3(c_1 c_2 - s_1 s_2) \\ l_2 s_1 + l_3(c_2 s_1 + c_1 s_2) \\ 1 \end{pmatrix}.$$

This solves for the coordinates of the hand. The direction of the hand can be computed as follows. The final direction of the hand will be $\theta_1 + \theta_2 + \theta_3 = \alpha$. Then let $c = \cos(\alpha)$ and $s = \sin(\alpha)$. Then by expanding the sum,

$$\begin{aligned} c &= c_1 c_2 c_3 - c_1 s_2 s_3 - s_1 c_2 s_3 - s_1 s_2 c_3 \\ s &= -s_1 s_2 s_3 + s_1 c_2 c_3 + c_1 s_2 c_3 + c_1 c_2 s_3. \end{aligned}$$

Now, we can get an explicit mapping $f : \mathcal{J} = V(x_1^2+y_1^2-1, x_2^2+y_2^2-1, x_3^2+y_3^2-1) \to \mathbb{R}^4$ as a function of $c_1, c_2, c_3, s_1, s_2, s_3$:

$$f(c_1, c_2, c_3, s_1, s_2, s_3) = \begin{pmatrix} l_2 c_1 + l_3(c_1 c_2 - s_1 s_2) \\ l_2 s_1 + l_3(c_2 s_1 + c_1 s_2) \\ c_1 c_2 c_3 - c_1 s_2 s_3 - s_1 c_2 s_3 - s_1 s_2 c_3 \\ -s_1 s_2 s_3 + s_1 c_2 c_3 + c_1 s_2 c_3 + c_1 c_2 s_3 \end{pmatrix}.$$

So far, we have only dealt with revolute joints, however we can easily extend the solution given in the previous example to accommodate prismatic joints.

Example 1.2.2. Consider the robot from Example 1.2.1, but let let there be an additional prismatic joint between segment 4 and the hand. Then $l_4 \in [m_1, m_2]$ for some $m_1, m_2 \in \mathbb{R}$. Then if we know the setting of l_4, then the position of the hand in (x_4, y_4) coordinates is $(l_4, 0)$. Therefore, the position of the hand can be found by:

$$\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = A_1 A_2 A_3 \begin{pmatrix} l_4 \\ 0 \\ 1 \end{pmatrix}.$$

The direction of the hand will be the same as in the previous problem, so now we have $f : \mathcal{J} = V(x_1^2 + y_1^2 - 1, x_2^2 + y_2^2 - 1, x_3^2 + y_3^2 - 1) \times [m_1, m_2] \to \mathbb{R}^4$ as a function of $c_1, c_2, c_3, s_1, s_2, s_3, l_4$:

$$f(c_1, c_2, c_3, s_1, s_2, s_3, l_4) = \begin{pmatrix} c_1 l_2 + l_3(c_1c_2 - s_1s_2) + l_4(c_1c_2c_3 - c_1s_2s_3 - s_1c_2s_3 - s_1s_2c_3) \\ l_2s_1 + l_3(c_2s_1 + c_1s_2) + l_4(-s_1s_2s_3 + s_1c_2c_3 + c_1s_2c_3 + c_1c_2s_3) \\ c_1c_2c_3 - c_1s_2s_3 - s_1c_2s_3 - s_1s_2c_3 \\ -s_1s_2s_3 + s_1c_2c_3 + c_1s_2c_3 + c_1c_2s_3 \end{pmatrix}.$$

If l_2 and l_3 had variable length, they could also become parameters of the function f.

The appendix will outline how to find f for a robot with n revolute joints.

1.3 Inverse Kinematic Problem

Given a point $(a, b) \in \mathbb{R}^2$ and an orientation $\phi \in [0, 2\pi]$, we wish to know if the robot hand can achieve this positioning, and what configuration of the joints will allow it to do so. In other words, if we have solved for $f : \mathcal{J} \to C$, given $c \in C$, what is $f^{-1}(c)$?

Consider the robot from Example 1.2.1, pictured in Figure 1.5.

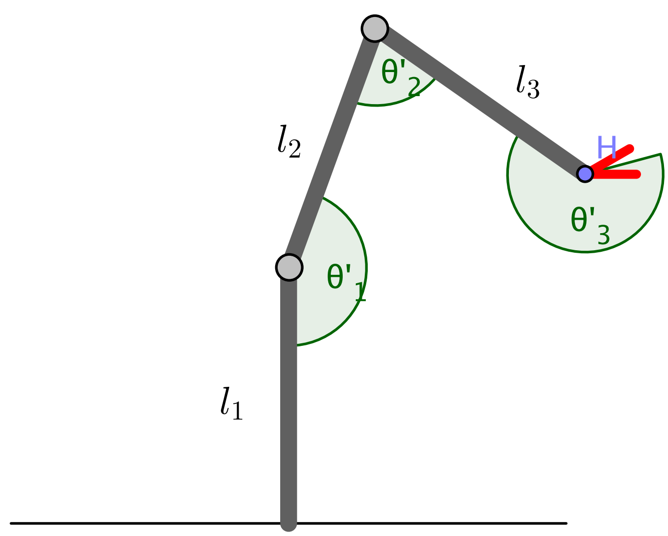
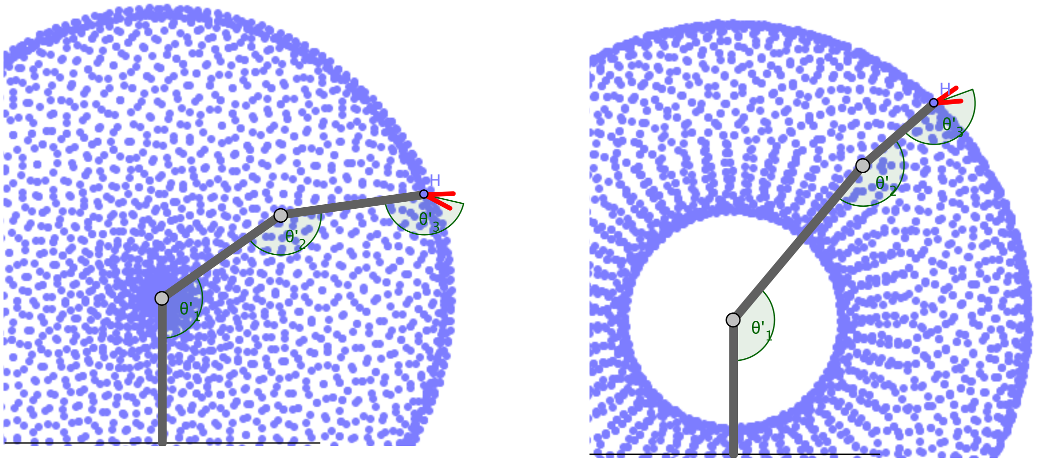


Figure 1.5: A robot with 3 revolute joints.

If $l_2 = l_3 = l$, then the hand can achieve any (a, b) in the disc of radius $2l$ centered at joint 1. If $l_2 \neq l_3$, this becomes an annulus (as in Figure 1.7). If the hand is directly connected to a revolute joint, it can achieve any orientation α by simply letting $\theta_3 = \alpha - \theta_1 - \theta_2$. Hence, we will primarily consider the inverse kinematic problem for the position of the hand, (a, b). Then from Example 1.2.1, the possible hand positions



(a) $l_2 = l_3$ (b) $l_2 > l_3$

Figure 1.6

are described by the system of polynomial equations (in c_1, s_1, c_2, and s_2):

$$\begin{aligned} a &= l_2c_1 + l_3(c_1c_2 - s_1s_2) \\ b &= l_2s_1 + l_3(c_2s_1 + c_1s_2) \\ 0 &= c_1^2 + s_1^2 - 1 \\ 0 &= c_2^2 + s_2^2 - 1 \end{aligned} \tag{1.3}$$

To solve this system of equations, we may simply compute a Groebner basis using lex order with $c_2 > s_2 > c_1 > s_1$. The solutions will depend on a, b, l_2, l_3, which are treated as coefficients of the Groebner basis. The reduced Groebner basis for the ideal $I \subseteq \mathbb{R}(a, b, l_2, l_3)[s_1, c_1, s_2, c_2]$, $I = (-a + l_2c_1 + l_3(c_1c_2 - s_1s_2), -b + l_2s_1 + l_3(c_2s_1 + c_1s_2), c_1^2 + s_1^2 - 1, c_2^2 + s_2^2 - 1)$ is

$$\begin{aligned} G = \Bigg\{ & c_2 - \frac{a^2 + b^2 - l_2^2 - l_3^2}{2l_2l_3}, \\ & s_2 + s_1 \cdot \frac{a^2 + b^2}{al_3} - \frac{a^2b + b^3 + b(l_2^2 - l_3^2)}{2al_2l_3}, \\ & c_1 + s_1 \cdot \frac{b}{a} - \frac{a^2 + b^2 + l_2^2 - l_3^2}{2al_2}, \\ & s_1^2 - s_1 \cdot \frac{a^2b + b^3 + b(l_2^2 - l_3^2)}{l_2(a^2 + b^2)} + \frac{(a^2 + b^2)^2 + (l_2^2 - l_3^2)^2 - 2a^2(l_2^2 + l_3^2) + 2b^2(l_2^2 - l_3^2)}{4l_2^2(a^2 + b^2)} \Bigg\} \end{aligned}$$

We must proceed with caution when working over $\mathbb{R}(a, b, l_2, l_3)$. In practice, we will simply substitute these variables with real numbers. This substitution creates an ideal $\bar{I} \subseteq \mathbb{R}[c_1, s_1, c_2, s_2]$ which corresponds to a specific configuration of the robot for the chosen values of a, b, l_2, l_3. Under this substitution, is G still a Groebner basis? Replacing variables by specific values over a field is called *specialization*. Then, we wish to know how a Groebner basis behaves under specialization. Glancing at the Groebner basis, it is reasonable to guess that we will encounter problems when the denominators vanish. Vanishing denominators are not the only thing that can go wrong in general (for this example, however, vanishing denominators are the only thing that can go wrong). It is fairly typical for the Groebner basis to behave nicely everywhere except on a couple of values of the parameters.

For this example, then, if $l_2, l_3 \neq 0$, $a \neq 0$, and $a^2+b^2 \neq 0$, then everything works out nicely and this Groebner basis is still a Groebner basis. By observing G, we can see that any s_1 which is a zero of the last polynomial in G can be extended to a full solution for the configuration of the robot. Furthermore, the solution set is finite, because the last equation is quadratic in s_1, so there are two solutions. One thing to consider is which a, b will give real solutions for s_1. Intuition based on geometry would indicate that we need $a^2+b^2 \leq (l_2+l_3)^2$.

Let $l_2 = l_3 = 1$, and substitute this into G, call this $\bar{G}$:

$$\bar{G} = \left\{ c_2 - \frac{a^2+b^2-2}{2}\ ,\ s_2 + s_1 \cdot \frac{a^2+b^2}{a} - \frac{a^2b+b^3}{2}, \right.$$
$$\left. c_1 + s_1 \cdot \frac{b}{a} - \frac{a^2+b^2}{2a},\ s_1^2 - bs_1 + \frac{(a^2+b^2)^2-4a^2}{4(a^2+b^2)} \right\}$$

Since the last element is quadratic in s_1, we can use the quadratic formula to solve for the possible values of s_1. This gives

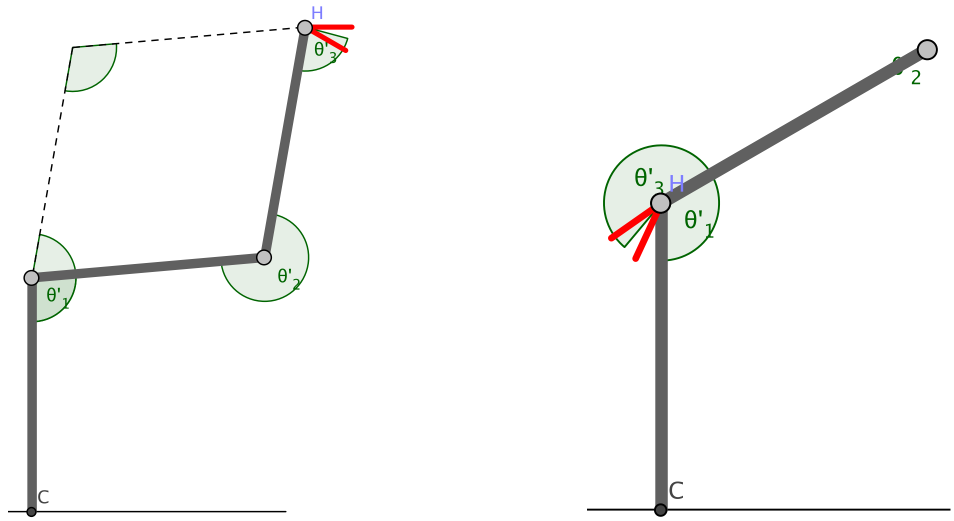
$$s_1 = \frac{b}{2} \pm \frac{|a|\sqrt{4-a^2-b^2}}{2\sqrt{a^2+b^2}}.$$

This only has real solutions when $a^2+b^2 \leq 4 = (l_2+l_3)^2$, which is what we would expect. For all $0 < a^2+b^2 < 4$, we have two solutions (see Figure 1.7a). Geometrically, we can see that if one solution has been found, with the configuration (θ_1, θ_2), then the other solution which will give the same configuration is given by $(\theta_1+\theta_2-\pi, 2\pi-\theta_2)$. When $a^2+b^2 = 4$, there is only one solution (which makes sense geometrically). From here, we could solve for the other variables.

In the case that $a = b = 0$, the hand is at the origin. Geometrically, this means that θ_1 can be any angle so long as $\theta_2 = \pi$, and this represents all of the solutions in this case (see Figure 1.7b).

If $a = 0$ and $b \neq 0$, we do not expect anything to go wrong. However, some denominators in the Groebner basis vanish. To fix this, we set $a = 0$ and $l_2 = l_3 = 1$ prior to computing the Groebner basis, and then recompute the Groebner basis.

In the cases where $a^2+b^2 = 0, 4$, unexpected things occurred. This would lead us to believe that something a little more is going on here. Let $f : \mathcal{J} \to C$ be the configuration mapping for some planar robot. Let the dimension of $\mathcal{J}$ $\dim(\mathcal{J})$ be defined as the number of degrees of freedom (in the previous example it would have been 3), and let $\dim(C)$ also represent the number of degrees of freedom (in the previous example, this would also be 3). Suppose that $\dim(\mathcal{J}) = m$ and $\dim(C) = n$. We have that f is differentiable, and the Jacobian matrix J_f for f will be an $n \times m$ matrix. Then if $j \in \mathcal{J}$ is substituted into J_f, we obtain the best linear approximation for f near j.



(a) Two possible solutions for the hand being in the same location in the $a \neq 0$ case.

(b) The case when $a = b = 0$.

Figure 1.7

We also know that $\text{rank}(J_f) \leq \min(m, n)$. If $\text{rank}(J_f) < \min(m, n)$, this indicates that there may be some singular behavior of f near j.

Definition A *kinematic singularity* is a point $j \in \mathcal{J}$ such that $J_f(j)$ has deficient rank.

Example 1.3.1. For the robot in the previous examples with three revolute joints and $l_2 = l_3 = 1$, the Jacobian is

$$J_f(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} -\sin(\theta_1+\theta_2) - \sin(\theta_1) & -\sin(\theta_1+\theta_2) & 0 \\ -\cos(\theta_1+\theta_2) - \cos(\theta_1) & -\cos(\theta_1+\theta_2) & 0 \\ 1 & 1 & 1. \end{pmatrix}$$

This matrix will have deficient rank iff the determinant is zero. So, we set

$$\begin{aligned} 0 &= \det(J_f) \\ &= \sin(\theta_1+\theta_2)\cos(\theta_1+\theta_2) + \sin(\theta_1)\cos(\theta_1+\theta_2) \\ &\quad - (\sin(\theta_1+\theta_2)\cos(\theta_1+\theta_2) + \cos(\theta_1)\sin(\theta_1+\theta_2)) \\ &= \sin(\theta_2). \end{aligned}$$

This implies that there are kinematic singularities when $\theta_2 = 0$ or when $\theta_2 = \pi$, which are the two special configurations we identified earlier.

Definition A robot is *kinematically redundant* if the dimension of $\mathcal{J}$ is larger than the dimension of the configuration space C.

Suppose we have a kinematically redundant robot with r revolute joints and $j \in \mathcal{J}$ is not a kinematic singularity. Then $f : \mathbb{R}^r \to \mathbb{R}^3$ is continuously differentiable, and suppose $f(j) = c$. Since j is not a kinematic singularity, the Jacobian of f at j has an invertible 3×3 submatrix. Let the variables corresponding to those in the matrix be (y_1, y_2, y_3), and let the remaining variables be the x_i, and let $j = (j_1, \ldots, j_{r-3}, k_1, k_2, k_3)$. Then by the implicit function theorem, there exists an open set $U \subseteq \mathbb{R}^{r-3}$ containing $(j_1, \ldots, j_{r-3})$ and an open subset $V \subseteq \mathbb{R}^3$ containing (k_1, k_2, k_3), and a continuously differentiable function $g : U \to V$ such that such that

$$\{(x, g(x))|x \in U)\} = \{(\mathbf{x}, \mathbf{y}) \in U \times V | f(\mathbf{x}, \mathbf{y}) = c\}.$$

Therefore, there are an infinite number of configurations which will allow the robot to be at c.

The last thing we may want is for the robot to follow a parametrized path $c(t) \in C$ starting from one point and ending at another point. To that end, we would need to find a path $j(t) \in \mathcal{J}$ such that $f(j(t)) = c(t)$. We may also want the path to have certain properties, such as $j(t)$ being a closed path (for repetitive tasks), a maximum joint speed, and a path with minimized joint movement.

Geometric Theorems

2.1 Automatic Geometric Theorem Proving

For many geometric theorems, we may introduce coordinates onto the Euclidean plane, and then restate the hypotheses and conclusions as polynomial equations. Then, algorithmic methods can be used to prove the theorems. Consider the following example:

Example 2.1.1. Let A, B, C, D be the vertices of a parallelogram in the plane. Then $\overline{AD}$ and $\overline{BC}$ intersect at a point which bisects each of them (see Figure 2.8). In this example, I will show how this can be restated as a system of polynomial equations. Let A be at the origin, and B be along the x axis. So $A = (0, 0)$ and $B = (u_1, 0)$ for

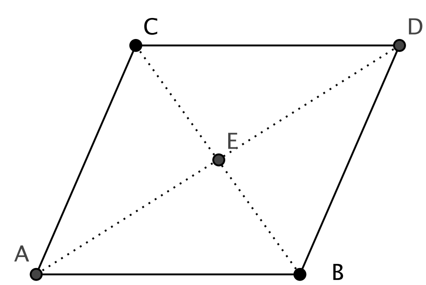


Figure 2.8

some $u_1 \neq 0$. Let $C = (u_2, u_3)$, where $u_3 \neq 0$. Then we could write the coordinates for D in terms of polynomial equations based on our assumption that $\overline{AC}$ is parallel to $\overline{BD}$ and that $\overline{CD}$ is parallel to $\overline{AB}$, but it is equivalent and simpler to just notice that $D = C + B = (u_1 + u_2, u_3)$. Now, we must find the coordinates of point E. Notice that E is collinear with A, D, and that it is also collinear with C, D. Let $E = (x_1, x_2)$. Then we get that

$$\begin{aligned} A, E, D \text{ collinear}: \ & \frac{x_2}{x_3} = \frac{u_3}{u_1 + u_2} \\ B, E, C \text{ collinear}: \ & \frac{x_2}{x_1 - u_1} = \frac{u_3}{u_2 - u_1}, \end{aligned}$$

And this gives the polynomial equations:

$$h_1 = x_2(u_1 + u_2) - x_3u_3 = 0$$
$$h_2 = x_2(u_2 - u_1) - u_3(x_1 - u_1) = 0.$$

This system of equations gives the hypotheses of the theorem. The conclusions can be stated using the pythagorean theorem:

$$AE = ED: \ x_1^2 + x_2^2 = (u_1 + u_2 - x_1)^2 + (u_3 - x_2)^2$$
$$CE = EB: \ (x_1 - u_2)^2 + (u_3 - x_2)^2 = (u_1 - x_1)^2 + x_2^2$$

So, the theorem states that if the polynomial equations corresponding to the hypotheses are true, then the polynomial equations stating the conclusion should be true.

Several geometric statements can be easily stated in terms of polynomial equations. Let A, B, C, D be points in the plane. Then the following represent ways to rewrite hypotheses as polynomial equations:

Hypothesis		Polynomial Equation
$\overline{AB}$ is parallel to $\overline{CD}$	$\Longrightarrow$	$\frac{b_2 - a_2}{b_1 - a_1} = \frac{d_2 - c_2}{d_1 - c_1}$
$\overline{AB}$ is perpendicular to $\overline{CD}$	$\Longrightarrow$	$\frac{b_2 - a_2}{b_1 - a_1} = -\frac{d_1 - c_1}{d_2 - c_2}$
A, B, C are collinear	$\Longrightarrow$	$\frac{b_2 - a_2}{b_1 - a_1} = \frac{c_2 - a_2}{c_1 - a_1}$
$AB = CD$	$\Longrightarrow$	$(b_2-a_2)^2+(b_1-a_1)^2=(d_2-c_2)^2+(d_1-c_1)^2$
C is on the circle at A with radius AB	$\Longrightarrow$	$AB = AC$
C is the midpoint of $\overline{AB}$	$\Longrightarrow$	A, B, C collinear and $AC = CB$

Definition A geometric theorem is *admissible* if both its hypotheses and conclusions can be translated into polynomial equations.

What is the typical form of an admissible geometric theorem? Let $u_1, \ldots, u_m$ be independent variables, and let $x_1, \ldots, x_n$ be dependent variables. Then the hypotheses will be some collection of polynomials in the u_i, x_j. It is typical for there to be n hypotheses, so we may write them as:

$$h_1(u_1, \ldots, u_m, x_1, \ldots, x_n) = 0$$
$$\vdots$$
$$h_n(u_1, \ldots, u_m, x_1, \ldots, x_n) = 0$$

and the conclusion may be written as

$$g(u_1, \ldots, u_m, x_1, \ldots, x_n) = 0.$$

Then g should vanish whenever the h_i do. This gives us the following definition.

Definition The conclusion g *follows strictly* from the hypotheses $h_1, \ldots, h_n$ if $g \in I(V) \subset \mathbb{R}[u_1, \ldots, u_m, x_1, \ldots, x_n]$, where $V = V(h_1, \ldots, h_n)$.

This seems like a reasonable definition, but it turns out to be too strict. Many geometric theorems have "degenerate" cases that this definition does not account for.

Proposition 2.1.1. If $g \in \sqrt{\langle h_1, \ldots, h_n \rangle}$, then g follows strictly from $h_1, \ldots, h_n$.

This follows directly from the fact that $\sqrt{I} \subseteq I(V(I))$. Since radical membership is computable, this is a useful proposition.

In certain cases, a geometric theorem may be true but the conclusion will not strictly follow from the hypotheses. In such cases, the variety decomposes into irreducible varieties, and degeneracies occur on some of the components of the decomposition, and on other components of the decomposition the conclusion g vanishes, as desired. Frequently when a degeneracy occurs, it is because an equation which is only dependent on one u_i holds on the variety, which is problematic because the u_i are independent variables. To that end, we will make the following definition.

Definition Let W be an irreducible variety in the affine space $\mathbb{R}^{m+n}$ with coordinates $u_1, \ldots, u_m, x_1, \ldots, x_n$. Then the $u_1, \ldots, u_m$ are *algebraically independent* on W if $I(W) \cap \mathbb{R}[u_1, \ldots, u_m] = \{0\}$.

Let V be written as a finite union of irreducible varieties as follows:

$$V = W_1 \cup \cdots \cup W_p \cup U_1 \cup \cdots \cup U_q,$$

where $u_1, \ldots, u_m$ are algebraically independent on the W_i components and not on the U_j components. To make sure that the u_i are actually arbitrary, we should only consider

$$V' = W_1 \cup \cdots \cup W_p.$$

Definition The conclusion g *follows generically* from the hypotheses $h_1, \ldots, h_n$ if $g \in I(V') \subseteq \mathbb{R}[u_1, \ldots, u_m, x_1, \ldots, x_n]$.

A geometric theorem is true if the conclusions follow generically from the hypotheses. How do we determine in as computationally easy way as possible whether $g \in I(V')$?

Proposition 2.1.2. We can say that g follows generically from $h_1, \ldots, h_n$ whenever there is some nonzero polynomial $c(u_1, \ldots, u_m)$ such that

$$c \cdot g \in \sqrt{H},$$

where H is the ideal generated by the hypotheses.

Proof. Suppose that there exists a c such that $c \cdot g \in \sqrt{H}$, and let V_j be one of the irreducible components of V'. Then $c \cdot g$ vanishes on $V_j \subseteq V$, so $c \cdot g \in I(V_j)$. Since V_j is irreducible, $I(V_j)$ is a prime ideal. This implies that either c or g is in $I(V_j)$, but $c \notin I(V_j)$ since the u_i are algebraically independent on V_j. Therefore, $g \in I(V_j)$. Since this is true for all j, we have that $g \in I(V')$. □

For this to be useful, we need to find a way of determining when there is a c such that $c \cdot g \in \sqrt{H}$. This holds iff

$$(c \cdot g)^s = \sum_{j=1}^{n} A_j h_j$$

for some $A_j \in \mathbb{R}[u_1, \ldots, u_m, x_1, \ldots, x_n]$. Dividing by c^s,

$$g^s = \sum_{j=1}^{n} \frac{A_j}{c^s} h_j,$$

so that $g \in \sqrt{\tilde{H}}$ generated by $h_1, \ldots, h_n$ over the ring $\mathbb{R}(u_1, \ldots, u_m)[x_1, \ldots, x_n]$. Conversely, if $g \in \sqrt{\tilde{H}}$, then

$$g^s = \sum_{j=1}^{n} B_j h_j.$$

Let c be the least common denominator for all the B_j, and multiply both sides by c^s (clearing the denominators),

$$(c \cdot g)^s = \sum_{j=1}^{n} B_j' h_j.$$

Then $c \cdot g \in \sqrt{H}$.

This gives an algorithmic method for proving that a conclusion follows generically from a set of hypotheses (because we know how to localize the relevant polynomial ring, and compute radical membership). Making the u_i invertible removes the degenerate cases. In the next section, we will look at another method for proving geometric theroems.

2.2 Wu's Method

A second method for algorithmically proving geometric theorems is Wu's method, which will be discussed here. First, we need something called *pseudodivision*.

Proposition 2.2.1. Let f and g be polynomials in the ring $k[x_1, \ldots, x_n, y]$ be written in the form

$$\begin{aligned} f &= c_p y^p + \cdots + c_1 y + c_0 \\ g &= d_m y^m + \cdots + d_1 y + d_0, \end{aligned}$$

where the c_i, d_j are polynomials in the x_k, $g \neq 0$, and $m \leq p$. Then there exist q and r such that

$$d_m^s f = qg + r,$$

where $q, r \in k[x_1, \ldots, x_n, y]$, and the degree of r in y is less than m. Furthermore, $r \in (f, g)$.

Algorithm: We may find q, r by employing the following algorithm, which is called *pseudodivision with respect to y*. Let deg and LC refer to the degree and leading coefficients of polynomials in y. Then

$$\begin{aligned} \text{input} &: f, g \\ \text{output} &: q, r \\ r &:= f; \\ g &:= 0 \\ \text{WHILE } r &\neq 0 \text{ AND } \deg(r) \geq m \text{ DO} \\ r &:= d_m r - LC(r) g y^{deg(r)-m} \\ r &:= d_m q + LC(r) y^{deg(r)-m}. \end{aligned}$$

The polynomials q and r are known as the *pseudoquotient* and the *pseudoremainder*. Let $Rem(f, g, y)$ be the pseudoremainder of the pseudodivision of f by g with respect to y.

Recall that we had $V = V' \cup U$, where V' is the union of the components on which the u_i are algebraically independent. We wish to show that g, the conclusion, vanishes on V'. I will present an elementary version of Wu's method in the case that V' is irreducible. There are two main steps:

- Use pseudodivision to reduce the hypotheses h_i to a set of polynomials f_i of the form:

  $$\begin{aligned} f_1 &= f_1(u_1, \ldots, u_m, x_1) \\ f_2 &= f_2(u_1, \ldots, u_m, x_1, x_2) \\ &\vdots \\ f_n &= f_n(u_1, \ldots, u_m, x_1, \ldots, x_n) \end{aligned}$$

  Such that $V(f_1, \ldots, f_n)$ contains V'.

- Use pseudodivision og g by successive f_i with respect to the variable x_i in order to determine whether $g \in I(V')$:

  $$\begin{aligned} R_{n-1} &= Rem(g, f_n, x_n) \\ R_{n-2} &= Rem(R_{n-1}, f_{n-1}, x_{n-1}) \\ &\vdots \\ R_0 &= Rem(R_1, f_1, x_1). \end{aligned}$$

First, I will discuss step 1. The process for step 1 goes one variable at a time, starting with x_n, as follows:

1. Let S be the set of all polynomials with the variable x_n
2. If there is only one polynomial in S, call it f_n, and then start over with $h_1, \ldots, h_{n-1}$ with respect to x_{n-1}.
3. If there is more than one polynomial in S, but one of them has degree 1 (say h_k) in x_n, then let $h_k = f_n$, and replace all other $h_i \in S$ with $Rem(h_i, h_k, x_n)$.
4. If there is more than one polynomial in S, and none has degree 1, do the following:
   (a) pick $a, b \in S$ where $0 < deg(b, x_n) \leq deg(a, x_n)$
   (b) compute $r = Rem(a, b, x_n)$
   (c) replace S by $(S \backslash \{a\}) \cup \{r\}$,

   and repeat this procedure until only one polynomial has x_n in it, and call this polynomial f_n.

Then, the process is repeated for each x_i as i decreases, until the polynomials are of triangular form. The triangular equations relate to the original hypotheses.

Proposition 2.2.2. Suppose that $f_1 = \cdots = f_n = 0$ are triangular equations obtained from $h_1 = \cdots = h_n = 0$ by the given algorithm. Then $V' \subset V \subset V(f_1, \ldots, f_n)$

Proof. We have that $(f_1, \ldots, f_n) \subset (h_1, \ldots h_n)$ by the previous proposition, so $V \subseteq V(f_1, \ldots, f_n)$, and we already know have that $V' \subset V$. $\square$

The following theorem completes Wu's method.

Theorem 2.2.1. Consider the set of hypotheses and the conclusion g for a geometric theorem. Let R_0 be the final remainder computed when completing successive pseudo-division using the triangular polynomials f_i. Let d_j be the leading coefficient on f_j as a polynomial in x_j. Then

(i) There are nonnegative integers $s_1, \ldots, s_n$ and polynomials $A_1, \ldots, A_n$ in the ring $\mathbb{R}[u_1, \ldots, u_m, x_1, \ldots, x_n]$ such that:

$$d_1^{s_1} \cdots d_n^{s_n} g = A_1 f_1 + \cdots + A_n f_n + R_0.$$

(ii) If R_0 is the zero polynomial, then g is zero at every point of $V' \backslash V(d_1 d_2 \cdots d_n) \subset \mathbb{R}^{m+n}$

Proof.

(i) This follows by applying the pseudodivision algorithm one polynomial and variable at a time:

$$\begin{aligned}
R_{n-1} &= d_n^{s_n} g - q_n f_n \\
&\lightning \\
R_{n-2} &= d_{n-1}^{s_{n-1}} d_n^{s_n} g - q_{n-1} f_{n-1} - d_{n-1}^{s_{n-1}} f_n \\
&\lightning \\
&\vdots \\
&\lightning \\
R_0 &= d_1^{s_1} \cdots d_n^{s_n} g - (A_1 f_1 + \cdots + A_n f_n)
\end{aligned}$$

(ii) By (i), if $R_0 = 0$ then $d_1^{s_1} \cdots d_n^{s_n} g = A_1 f_1 + \cdots + A_n f_n$, so the left hand side and the right hand side vanish on the same points, $V(f_1, \ldots, f_n)$. Then either g or one of the $d_j^{s_j}$ vanishes on this variety. Since $V' \subseteq V(f_1, \ldots, f_n)$, one of them vanishes on V'. Therefore, we must have that g is zero on every point of $V' \backslash V(d_1 d_2 \cdots d_n)$.

$\square$

This version of Wu's method only gives $g = 0$ under the condition that $d_j \neq 0$. This can lead to problems, especially when V' is reducible. Stronger versions of this theorem are known. The attached *Mathematica* file implements Wu's method and uses it to prove the Circle Theorem of Apollonius.

Appendix

This section will solve the forward kinematic problem for a robot with n revolute joints. Consider a planar robot with a fixed segment 1, and with n revolute joints linking the segments of length $l_2, \ldots, l_n$. The hand is segment $n+1$, and is attached to segment n by joint n. Then we have that $\mathcal{J} = (S^1)^n$ and $C = \mathbb{R}^2 \times S^1$. The mapping $f : \mathcal{J} \to C$ is given by

$$f(\theta_1, \ldots, \theta_n) = \begin{pmatrix} \sum_{i=1}^{n-1} l_{i+1} \cos\left(\sum_{j=1}^{i} \theta_j\right) \\ \sum_{i=1}^{n-1} l_{i+1} \sin\left(\sum_{j=1}^{i} \theta_j\right) \\ \sum_{j=1}^{n} \theta_j \end{pmatrix}.$$

Proof. The third entry of this vector is clear, because this represents the orientation of the hand, and the orientation of the hand is simply given by the sum of the angles of the revolute joints. I will proceed with the first two entries of the vector by induction.

Suppose $n = 1$. Then

$$\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{0} l_{i+1} \cos\left(\sum_{j=1}^{i} \theta_j\right) \\ \sum_{i=1}^{0} l_{i+1} \sin\left(\sum_{j=1}^{i} \theta_j\right) \\ 1 \end{pmatrix}.$$

Now, suppose that for a robot with $n-1$ revolute joints, we know that

$$\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = A_1 A_2 \cdots A_{n-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n-2} l_{i+1} \cos\left(\sum_{j=1}^{i} \theta_j\right) \\ \sum_{i=1}^{n-2} l_{i+1} \sin\left(\sum_{j=1}^{i} \theta_j\right) \\ 1 \end{pmatrix}$$

Knowing that

$$A_1 = \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Some algebra and trigonometric identities will yield

$$\begin{pmatrix} \sum_{i=1}^{n-2} l_{i+1} \cos\left(\sum_{j=1}^{i} \theta_j\right) \\ \sum_{i=1}^{n-2} l_{i+1} \sin\left(\sum_{j=1}^{i} \theta_j\right) \\ 1 \end{pmatrix} = A_1 \begin{pmatrix} \sum_{i=1}^{n-2} l_{i+1} \cos\left(\sum_{j=2}^{i} \theta_j\right) \\ \sum_{i=1}^{n-2} l_{i+1} \sin\left(\sum_{j=2}^{i} \theta_j\right) \\ 1 \end{pmatrix} = A_1 (A_2 \cdots A_{n-1}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So, by reindexing the previous (to find the vector that gives the product of $A_3 \cdots A_n$),

$$
\begin{aligned}
\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} &= A_1 A_2 \cdots A_{n-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= A_1 A_2 \begin{pmatrix} \sum_{i=2}^{n-1} l_{i+1} \cos\left(\sum_{j=3}^{i} \theta_j\right) \\ \sum_{i=2}^{n-1} l_{i+1} \sin\left(\sum_{j=3}^{i} \theta_j\right) \\ 1 \end{pmatrix} \\
&= A_1 \begin{pmatrix} \cos(\theta_2)\left[\sum_{i=2}^{n-1} l_{i+1} \cos\left(\sum_{j=3}^{i} \theta_j\right)\right] - \sin(\theta_2)\left[\sum_{i=2}^{n-1} l_{i+1} \sin\left(\sum_{j=3}^{i} \theta_j\right)\right] + l_2 \\ \sin(\theta_2)\left[\sum_{i=2}^{n-1} l_{i+1} \cos\left(\sum_{j=3}^{i} \theta_j\right)\right] - \cos(\theta_2)\left[\sum_{i=2}^{n-1} l_{i+1} \sin\left(\sum_{j=3}^{i} \theta_j\right)\right] \\ 1 \end{pmatrix} \\
&= A_1 \begin{pmatrix} \sum_{i=2}^{n-1} l_{i+1} \cos\left(\sum_{j=2}^{i} \theta_j\right) + l_2 \cos\left(\sum_{j=2}^{1} \theta_j\right) \\ \sum_{i=2}^{n-1} l_{i+1} \sin\left(\sum_{j=2}^{i} \theta_j\right) + l_2 \sin\left(\sum_{j=2}^{1} \theta_j\right) \\ 1 \end{pmatrix} \\
&= A_1 \begin{pmatrix} \sum_{i=1}^{n-1} l_{i+1} \cos\left(\sum_{j=2}^{i} \theta_j\right) \\ \sum_{i=1}^{n-1} l_{i+1} \sin\left(\sum_{j=2}^{i} \theta_j\right) \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta_1)\left[\sum_{i=1}^{n-1} l_{i+1} \cos\left(\sum_{j=2}^{i} \theta_j\right)\right] - \sin(\theta_1)\left[\sum_{i=1}^{n-1} l_{i+1} \sin\left(\sum_{j=2}^{i} \theta_j\right)\right] \\ \sin(\theta_1)\left[\sum_{i=1}^{n-1} l_{i+1} \cos\left(\sum_{j=2}^{i} \theta_j\right)\right] - \cos(\theta_1)\left[\sum_{i=1}^{n-1} l_{i+1} \sin\left(\sum_{j=2}^{i} \theta_j\right)\right] \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \sum_{i=1}^{n-1} l_{i+1} \cos\left(\sum_{j=1}^{i} \theta_j\right) \\ \sum_{i=1}^{n-1} l_{i+1} \sin\left(\sum_{j=1}^{i} \theta_j\right) \\ 1 \end{pmatrix}
\end{aligned}
$$

$\square$