AN ALTERNATIVE PROOF OF A THEOREM BY GESSEL

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In what follows we study a formula of Gessel which gives an interpretation for the Tutte polynomial $T_G(x, y)$ evaluated at $x = 1$. We will prove the formula using deletion-contraction techniques often appearing with the Tutte polynomial.

Theorem 1 (Gessel [Ges95])**.** *Let G be a connected graph on the vertices $\{1, \ldots, n\}$ with distinguished vertex $v = n$. If $V_i \subset V(G)\backslash\{v\}$, then let $G[V_i]$ be the induced subgraph of G at V_i. Let $\epsilon(V_i)$ be the number of edges from vertices in V_i to v. Then*

$$I_G(y) = \sum_{V_1,\ldots,V_k} \prod_{i=1}^{k} \left(1 + y + \cdots + y^{\epsilon(V_i)-1}\right) I_{G[V_i]}(y), \tag{1}$$

Where the sum is over all partitions $\{V_1, \ldots, V_k\}$ of $V(G)\backslash\{v\}$ such that $G[V_i]$ is connected for all i.

Proof. We may compute $I_G(y)$ by performing the deletion-contraction process for computing the Tutte polynomial, and replacing all x's with 1's. We will delete and contract edges of G in the following way:

1. Only delete or contract edges which do not contain the vertex n.
2. When an edge $\{i, j\}$ is contracted, label the resulting vertex $\{i, j\}$.

We do the second step in order to bookkeep which vertices were joined together. This is because in the end, each graph will correspond to a partition of $V\backslash\{n\}$. Once there are no more edges which do not contain the vertex n which can be deleted or contracted, stop. Graphs at the end of each path in the computation will be of the form in Figure 1. We denote by B_i the graph with two

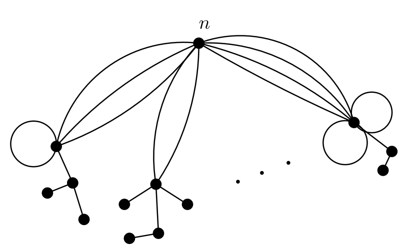

Figure 1: The graphs at the end of the deletion-contraction process described in the proof.

vertices and i edges between them. Then, the graphs obtained at the end of our computation are a sequence of graphs $B_{i_1}, \ldots, B_{i_s}$ joined together at the vertex n with trees and loops at the other vertex.

If n_j is the number of loops attached to one end of B_{i_j}, then the $I(y)$ polynomial of this graph is

$$\prod_{j=1}^{s} I_{B_{i_j}}(y) y^{n_j} = \prod_{j=1}^{s} (1 + y + \cdots + y^{i_j-1}) y^{n_j}.$$

Then, by the deletion-contraction formulation of the Tutte polynomial, $I_G(y)$ is the sum of these. So, we investigate these polynomials more closely.

For each graph H at the end of a path in the computation of the Tutte polynomial, associate to it a partition $\{V_1, \ldots, V_k\}$ of $V \backslash \{v\}$ in the following way. Remove the vertex n from the graph. Then take the union of the labels on all vertices in any single connected component of the resulting graph, and let this be one part in the partition. To denote this, we will write $H \sim \{V_1, \ldots, V_k\}$.

Then,

$$\begin{aligned}
I_G(y) &= \sum_{\{V_1,\ldots,V_k\}} \sum_{H\sim\{V_1,\ldots,V_k\}} I_H(y) \\
&= \sum_{\{V_1,\ldots,V_k\}} \sum_{H\sim\{V_1,\ldots,V_k\}} \prod_{i=1}^{k} (1 + y + \cdots + y^{\epsilon(V_i)-1}) I_{H[V_i]}(y) \\
&= \sum_{\{V_1,\ldots,V_k\}} \sum_{H\sim\{V_1,\ldots,V_k\}} \left(\prod_{i=1}^{k} (1 + y + \cdots + y^{\epsilon(V_i)-1}) \right) \left(\prod_{i=1}^{k} I_{H[V_i]}(y) \right) \\
&= \sum_{\{V_1,\ldots,V_k\}} \left(\prod_{i=1}^{k} (1 + y + \cdots + y^{\epsilon(V_i)-1}) \right) \left(\sum_{H\sim\{V_1,\ldots,V_k\}} \prod_{i=1}^{k} I_{H[V_i]}(y) \right).
\end{aligned}$$

Now, we claim that

$$\sum_{H\sim\{V_1,\ldots,V_k\}} \prod_{i=1}^{k} I_{H[V_i]}(y) = \prod_{i=1}^{k} I_{G[V_i]}(y),$$

and this will complete the proof. First notice that

$$\sum_{H\sim\{V_1,\ldots,V_k\}} \prod_{i=1}^{k} I_{H[V_i]}(y) = \prod_{i=1}^{k} \sum_{H[V_i]} I_{H[V_i]}(y),$$

where the sum on the right is over possible $H[V_i]$ such that $H \sim \{V_1, \ldots, V_k\}$, and the sum contains no duplicate $H[V_i]$. We interpret this in the following way. In order to create an H with $H \sim \{V_1, \ldots, V_k\}$, one must select one of a number of possible $H[V_i]$ for each $1 \leq i \leq k$. Furthermore, every H created in this way is indeed possible.

Then it remains to show that

$$\sum_{H[V_i]} I_{H[V_i]}(y) = I_{G[V_i]}(y).$$

To do this, we consider what happens in the branches of the Tutte computation that give graphs corresponding to the partition $\{V_1, \ldots, V_k\}$. To obtain $\{V_1, \ldots, V_k\}$, all edges from a vertex of V_i to a vertex of V_j must be deleted. From that point, I claim that the possible deletions and contractions which preserve the partition are exactly those which are either also legal in $G[V_i]$ or do not alter the polynomial $I(y)$. If an edge e is in $G[V_i]$, then one of several things may happen.

1. It is not legal to delete or contract e.

2. It is legal to delete or contract e in G and in $G[V_i]$.

3. It is legal to delete or contract e in G, but not in $G[V_i]$.

The first and second cases give no problems. In the third case, we must consider what happens when we decide to delete and contract e. The path in which we delete e alters the partition, so we ignore this path when considering those corresponding to the partition $\{V_1, \ldots, V_k\}$. On the other hand, if it is not legal to contract e in $G[V_i]$, this means that e is a bridge in $G[V_i]$, and so contracting it will make no new loops. Since loops are the only contributors to $I(y)$, contracting e does not affect the computation.

The set of moves which are legal in $G[V_i]$ will always be legal in G. So, we have shown that the paths in the computation of $I_G(y)$ which preserve the partition $\{V_1, \ldots, V_k\}$ are in direct correspondence with the paths of the computation of $I_{G[V_1]\cup\cdots\cup G[V_k]}(y)$. Furthermore, contributions from the graphs in each path in both computations will be the same. Hence,

$$\sum_{H[V_i]} I_{H[V_i]}(y) = I_{G[V_i]}(y).$$

This completes the proof. □

Example. We now give an example to illustrate how the deletion-contraction process works in this proof. Let G be the graph depicted at the top of Figure 2.

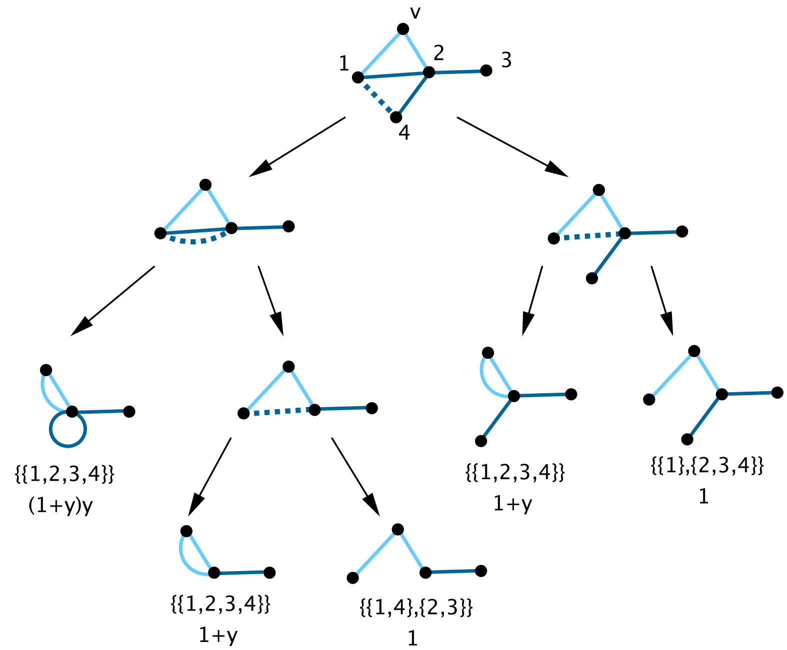

Figure 2: Computation of the Tutte polynomial using no edges connected to v. In the end, each graph is labeled with the partition $\{V_1, \ldots, V_2\}$ that it corresponds to, as well as its contribution to $I_G(y)$.

After completing the deletion-contraction process ignoring any edges connected to v, if we add together all $I(y)$ which correspond to the same partition, we get the corresponding component in the sum of Gessel's formula. So, we have:

Partition of $V \backslash \{v\}$	Contribution to $I_G(y)$
$\{\{1,2,3,4\}\}$	$(1+y)(y+2)$
$\{\{1,4\},\{2,3\}\}$	1
$\{\{1\},\{2,3,4\}\}$	1

For each partition $\{V_1,\ldots,V_k\}$, these are precisely the $\prod_{i=1}^{k}\left(1+y+\cdots+y^{\epsilon(V_i)-1}\right) I_{G[V_i]}(y)$, so we sum them to obtain $I_G(y)$.

References

[Ges95] Ira M. Gessel. Enumerative applications of a decomposition for graphs and digraphs. *Discrete Mathematics*, 139(1–3):257 − 271, 1995.