

# Revision Notes: Analytic Topology. (c1/2)

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FHS Part (II)

## Notation:

Lectures are Dr. Knight's 2003 series.

Willard refers to Willard, S. *General Topology*.

## 1 Separation Axioms.

Lectures: 1-3; Willard: pp. 85-102.

The *Separation Axioms* are variations on the Hausdorff property.

A space is  $T_0$  iff given any distinct points  $x, y$  there is an open set  $U$  which contains one and not the other.

A space is  $T_1$  iff given any distinct points  $x, y$  there is an open set  $U$  which contains  $x$  but not  $y$ .

A space is  $T_2$  (*Hausdorff*) iff given any distinct points  $x, y$  there are disjoint open sets  $U, V$  such that  $x \in U, y \in V$ .

A space,  $X$ , is *regular* iff closed subset  $F \subset X$  and a point  $x \notin F$  there are disjoint open sets  $U, V$  such that  $x \in U, F \subseteq V$ .  $X$  is  $T_3$  iff it's  $T_1$  and regular.

A space,  $X$ , is *completely regular* iff for  $F, x$  as above,  $\{x\}$  and  $F$  are *functionally separated*, that is there is a continuous function  $f : X \rightarrow [0, 1]$  such that  $f[\{x\}] \subseteq \{1\}$ ,  $f[F] \subseteq \{0\}$ .  $X$  is  $T_{3\frac{1}{2}}$ , or *Tychonoff*, iff it's  $T_1$  and completely regular.

$X$  is *normal* iff given any two disjoint closed subsets,  $F, G \subseteq X$ , there are disjoint open subsets of  $X, U, V$ , such that  $F \subseteq U, G \subseteq V$ .  $X$  is  $T_4$  iff it's  $T_1$  and normal.

In the presence of compactness,  $T_2, T_3$  and  $T_4$  are equivalent.

All the axiom  $T_i$  are closed under subsets and products (in fact a product is  $T_i$  iff each factor is) for for  $i < 4$ .  $T_4$  is closed under taking closed subsets, but not under (even finite) products. None of them are closed under taking quotients. Subspaces of  $T_4$  spaces are  $T_{3\frac{1}{2}}$ .

**Lemma 1.1 (Equivalent definitions).**  $X$  is  $T_1$  iff every singleton is closed.  $X$  is normal iff for every closed  $F$  and open  $U$  satisfying  $F \subseteq U$  there is an open  $V$  such that  $C \subseteq V \subset \bar{V} \subseteq U$ .

**Theorem 1.2 (Urysohn's Lemma).** Let  $F, G \subseteq X$  be disjoint closed subsets of a normal space,  $X$ . Then,  $C, D$  are functionally separated.

*Proof.* Let  $\mathbb{Q} \cap [0, 1] = \{r_n : n \in \mathbb{N}\}$  with  $r_0 = 1, r_1 = 0$ . We construct a family,  $(U_{r_n})$ , of open subsets of  $X$  by recursion on  $n$  with the property that  $r_m < r_n \rightarrow \bar{U}_{r_m} \subseteq U_{r_n}$ .

$F \subseteq X \setminus G$  which is open, so there is an open  $U_1$  such that  $F \subseteq U_1 \subseteq \bar{U}_1 \subseteq X \setminus G$ . Hence, there is also a  $U_0$  satisfying  $F \subseteq U_0 \subseteq \bar{U}_0 \subseteq U_1$ .

Now (inductive step), suppose  $U_1, U_0, U_{r_2}, \dots, U_{r_n}$  have been constructed. Let  $a_n$  be the member of  $\{0, 1, r_2, \dots, r_n\}$  next before  $r_{n+1}$  and  $b_n$  the member next after it. Then let  $U_{n+1}$  be some set satisfying  $\overline{U_{a_n}} \subseteq U_{r_{n+1}} \subseteq \overline{U_{r_{n+1}}} \subseteq U_{b_n}$ .

We now define  $f$  thusly:

$$f(x) := \inf(\{1\} \cup \{r \in \mathbb{Q} \cap [0, 1] : x \in U_r\})$$

Then,  $f$  is indeed a function from  $X$  to  $[0, 1]$  which maps  $F$  into 0 and  $G$  into 1. It remains to show it continuous. Now,  $\{(-\infty, q) : q \in \mathbb{Q}\} \cup \{(q, \infty) : q \in \mathbb{Q}\}$  is a sub-basis for  $\mathbb{R}$ , so we need only check that the pre-image of each such interval is open to show  $f$  continuous.

We start with  $f^{-1}(-\infty, q)$ .  $x \in f^{-1}(-\infty, q)$  (examining the definition) either  $q > 1$  (in which case we're fine) or there's an  $r < q$  such that  $x \in U_r \subseteq f^{-1}(-\infty, q)$ .

Now, for  $f^{-1}(q, \infty)$ . If  $x \in f^{-1}(q, \infty)$  then  $q < 1$  and either  $q < 0$  (in which case we're fine) or  $q \geq 0$  and there is some  $r > q$  such that  $x \notin U_r$ . Let  $s \in \mathbb{Q} \cap (q, r)$ . Then,  $U_s \subseteq \overline{U_s} \subseteq U_r$ , so  $x \in X \setminus \overline{U_s}$  which is open and  $\overline{U_s} \subseteq f^{-1}(q, \infty)$ .  $\square$

## 2 Products and convergence.

Lectures: 3-5; Willard: pp. 52-57, 77-81, 120.

Let  $\{X_\lambda : \lambda \in \Lambda\}$  be a family of sets.  $\prod_{\lambda \in \Lambda} X_\lambda$  is the set of all functions,  $f$ , with domain  $\Lambda$  such that for all  $\lambda \in \Lambda$ ,  $f(\lambda) \in X_\lambda$ . We define the *Tychonoff topology* on this product to be the coarsest topology that makes the projection mappings continuous. Equivalently this is the topology generated by sets of the form

$$U_\lambda \times \prod_{\mu \neq \lambda} X_\mu$$

where  $U_\lambda$  is an open subset of  $X_\lambda$ .

We want to show that a product is compact iff each factor is. It will be convenient to work with a definition of compactness which mirrors the definition possible in metric spaces that a space is compact iff every sequence has a convergent subsequence. This sequential-definition doesn't work in general, but a similar one using filters does.

A *filter* on set  $X$  is a subset of  $\mathcal{P}(X)$  satisfying

1.  $\emptyset \notin \mathcal{F}$ ;  $X \in \mathcal{F}$
2.  $\mathcal{F}$  is closed under finite intersections
3.  $\mathcal{F}$  is closed under taking supersets.

For  $x \in X$ , the *neighbourhood filter about  $x$* ,  $\mathcal{N}_x$ , is the set of all  $N \subset X$  such that  $x \in N^\circ$ .  $\mathcal{F}$  converges to  $x$  iff  $\mathcal{N}_x \subseteq \mathcal{F}$ .

We can think of filters as corresponding to sequences by a filter being a collection of sets that contain all but an initial segment of a sequence. Then, moving to a subsequence actually corresponds to moving to a super-filter. An *ultrafilter* is a filter,  $\mathcal{U}$  for which, for all  $A \subseteq X$ ,  $A \in \mathcal{U}$  or  $X \setminus A \in \mathcal{U}$ . Ultrafilters are maximal filters and all filters may be embedded in an ultrafilter.

**Theorem 2.1.** *If every ultrafilter on  $X$  converges, then  $X$  is compact.*

*Proof.* We prove the contrapositive.

Let  $\{V_\iota : \iota \in I\}$  be an open cover with no finite subcover. Define a filter,

$$\mathcal{F} := \{X \setminus \bigcup_{j=1}^n V_{\iota_j} : n \in \mathbb{N}, \iota_1, \dots, \iota_n \in I\}$$

[check it is a filter]. Extend  $\mathcal{F}$  to an ultra filter,  $\mathcal{U}$ . Suppose  $\mathcal{U} \rightarrow x$ . Then  $\mathcal{N}_x \subseteq \mathcal{U}$ . The  $(V_\iota)$  form a cover, so there is some  $V_n \ni x$ . Then  $V_n \in \mathcal{N}_x$ , so  $V_n \in \mathcal{U}$ . But,  $X \setminus V_n \in \mathcal{F}$ , so  $X \setminus V_n \in \mathcal{U}$ . So, their intersection,  $\emptyset \in \mathcal{U}$ . Contradiction.  $\square$

We may now prove our result on compactness.

**Theorem 2.2 (Tychonoff's theorem).**  $\prod_{\lambda \in \Lambda} X_\lambda$  is compact iff each factor is.

*Proof.* Only if is easy (continuous image of a compact space is compact).

For the if part, we show that all ultrafilters converge on the product. Let  $\mathcal{U}$  be an ultrafilter on  $\prod_{\lambda \in \Lambda} X_\lambda$ . Let  $\pi_\lambda$  be the projection onto the  $\lambda^{\text{th}}$  coordinate. It is easy to show that for each  $\lambda$ ,  $\pi_\lambda(\mathcal{U})$  is an ultrafilter on  $X_\lambda$ , which is compact, so  $\pi_\lambda(\mathcal{U})$  converges, to  $x_\lambda$ , say. Let  $f \in \prod_{\lambda \in \Lambda} X_\lambda$  be such that  $f(\lambda) = x_\lambda$  for all  $\lambda \in \Lambda$ . We want to show  $\mathcal{U} \rightarrow f$ , so we must show that  $\mathcal{N}_f \subseteq \mathcal{U}$ .

Let  $N \in \mathcal{N}_f$ , say  $N \subseteq \prod_{\lambda \in \Lambda} U_\lambda$ , a basic open set. For  $\lambda \neq \mu$ , say, we must have  $U_\lambda = X_\lambda$ , so  $X_\lambda \in \pi_\lambda(\mathcal{U})$ .

Now,  $\pi_\mu(\mathcal{U}) \rightarrow x_\mu$ , so  $x_\mu \in U_\mu \in \pi_\mu(\mathcal{U})$ . So  $U_\mu = \pi_\mu(V)$  for some  $V \in \mathcal{U}$ . So,  $V \times \prod_{\lambda \neq \mu} X_\lambda \in \mathcal{U}$ , so  $N \in \mathcal{U}$ . This does it.  $\square$

A product is locally compact iff each of its factors is compact, except for finitely many which are still locally compact.

### 3 Compactifications

Lectures: 6-8; Willard: pp. 135-40.

We wish to see what kinds of spaces may be embedded in compact  $T_2$  spaces. Any compact  $T_2$  space is  $T_4$ , so any subspace will have to be  $T_{3\frac{1}{2}}$ . Formally, we define a *Hausdorff-compactification* to be a pair  $\langle h, Y \rangle$  where  $Y$  is a compact  $T_2$  space and  $h : X \rightarrow Y$  is a map satisfying

1.  $h$  is an injection.
2.  $h : X \rightarrow h(X)$  is a homeomorphism
3.  $h(X)$  is dense in  $Y$ .

Informally, we think of  $X$  as a subspace of  $Y$ .

If  $\langle h, \gamma X \rangle$  and  $\langle k, \delta X \rangle$  are Hausdorff compactifications of  $X$  we say that  $\gamma X \leq \delta X$  iff there is a continuous  $f : \delta X \rightarrow \gamma X$  such that  $h = f \circ k$ . That is, the following diagram commutes:

$$\begin{array}{ccc} \gamma X & \xleftarrow{f} & \delta X \\ & h \searrow X \nearrow k & \end{array}$$

### 3.1 The Alexandroff one-point compactification

We say that a space,  $X$ , is *locally compact* iff for all  $x \in X$  and open  $U \ni x$  there is an open  $V$  and a compact  $K$  such that  $x \in V \subseteq K \subseteq U$ . Equivalently, each point in  $X$  has an open neighbourhood whose closure is compact.

For  $X$ , a locally compact  $T_2$  space which isn't compact, we define the *Alexandroff one-point compactification* to be the pair  $\langle h, \alpha X \rangle$  where  $\alpha X := X \cup \{\star\}$  with the topology whose open sets consists of

1.  $U \subseteq X$  which are open in  $X$
2.  $U \ni \star$  such that  $X \setminus U$  is compact.

$h$  is the identity.

**Proposition 3.1.** *The Alexandroff one-point compactification is indeed a  $T_2$  compactification.*

*Proof.* Hausdorffness: Easy if  $x, y \in X$ . To separate  $x, \star$  pick some open neighbourhood of  $x$  which is contained in  $X$  and whose closure is compact. The complement of this closure is an open neighbourhood of  $\star$ .

Compactness: Given an open cover, pick some set containing  $\star$ . The complement of this is compact.

$h$  is obviously an injection, and  $X$  isn't compact so  $\{\star\}$  is not open, so  $X$  is dense in  $\alpha X$ .

$h^{-1}$  continuous is easy. To show  $h$  continuous, note that  $h^{-1}(V) = V \cap X$  which, if  $\star \in V$  is open as  $X \setminus V$  is a compact subspace of a Hausdorff space, so closed.  $\square$

**Theorem 3.2 (Minimality).** *The Alexandroff Hausdorff compactification of is minimal.*

*Proof.* Let  $\langle k, \delta X \rangle$  be some other  $T_2$  compactification. Define  $f : \delta X \rightarrow \alpha X$  by:

1.  $f(k(x)) = h(x)$
2. if  $y \in \delta X \setminus k(X)$ ,  $f(y) = \star$ .

This  $f$  clearly makes the right diagram commute.

Continuity: Suppose  $\star \in U$ , an open subset of  $\alpha X$ . Then,  $f^{-1}(U) = \delta X \setminus k(X \setminus U)$ . But,  $k(X \setminus U)$  is compact and so closed, so  $f^{-1}(U)$  is open.

The proof that  $f^{-1}(U)$  is open when  $\star \notin U$  requires the following fact whose proof is long and uninspiring: for any  $T_2$  compactification of a locally compact space,  $X$  is open in  $\delta X$ . It is then easy.  $\square$

We say a continuous surjection  $f : X \rightarrow Y$  is *proper* iff it is closed and has compact fibres. (Fibres are level sets, ie. sets of the form  $f^{-1}(y)$  for  $y \in Y$ ).

**Theorem 3.3 (Extending maps).** *If  $X$  is  $T_2$ , locally compact and  $f : X \rightarrow Y$  (then so is  $Y$ ) and we can extend  $f$  to a proper map  $g : \alpha X \rightarrow \alpha Y$ .*

*Proof.*  $g := f \cup \{\langle \star_X, \star_Y \rangle\}$ .

That  $g$  is onto is obvious.

Continuity: If  $U \subseteq Y$  is open, then  $f^{-1}(U) = g^{-1}(U)$  which is open in  $X$  which is open in  $\alpha X$ . If  $\star_y \in U$ ,  $U$  open, then  $\star_X \in f^{-1}(U)$  and  $X \setminus g^{-1}(U) = f^{-1}(Y \setminus U)$  which (using properness) may be shown compact.

$g$  is closed as  $\alpha X$  and  $\alpha Y$  are both compact, so their closed are the same as their compact subsets.

Compact fibres is easy. □

### 3.2 The Stone-Čech compactification.

That last theorem about proper functions does not generalise to continuous functions using the Alexandroff compactification, which is one of the reason we introduce another compactification.

Given a  $T_{3\frac{1}{2}}$  space,  $X$ , let  $\{f_\lambda : \lambda \in \Lambda\}$  be the set of all continuous bounded functions from  $X$  to  $\mathbb{R}$ ,  $f_\lambda$  bounded by  $I_\lambda = [a_\lambda, b_\lambda]$ , say. Let  $Y$  be the Tychonoff product of the  $I_\lambda$ s and define  $h : X \rightarrow Y$  by  $h(x) : \lambda \mapsto f_\lambda(x)$ ,  $\beta X := \overline{h(X)}^Y$ .  $\langle h, \beta X \rangle$  is called the *Stone-Čech compactification* of  $X$ .

**Proposition 3.4.** *The Stone-Čech compactification is indeed a  $T_2$  compactification.*

*Proof.* It's 1 : 1 because, as  $X$  is  $T_{3\frac{1}{2}}$ , singletons are functionally separated.

One may check it's continuous by looking at the pre-image of some sub-basic open set.

Compactness of  $\beta X$  and density of  $X$  are easy.

To show  $h^{-1}$  open, we may show  $h$  closed. Let  $U \subset X$  be open. WLOG,  $U$  is non-empty, say  $x \in U$ .  $X$  is  $T_{3\frac{1}{2}}$ , so  $x$  is functionally separated from  $X \setminus U$  by, say,  $f_\lambda$  (say  $f_\lambda(x) = 1$ ,  $f_\lambda[X \setminus U] \subseteq \{1\}$ ). Let  $V := f_\lambda^{-1}[0, 1)$ . Then  $V$  is open and  $x \in V \subseteq U$ .

$$\begin{aligned} h(V) &= \{h(x) : f_\lambda \in [0, 1)\} \\ &= \{h(x) : h(x)(\lambda) \in [0, 1)\} \\ &= \{y \in h(X) : y(\lambda) \in [0, 1)\} \\ &= h(X) \cap [0, 1)_\lambda \times \prod_{\mu \neq \lambda} I_\mu \end{aligned}$$

This is open, so we're done. □

The Stone-Čech compactification is characterised by the Stone-Čech property. We prove this in the next two theorems.

**Theorem 3.5 (Stone-Čech property).** *If  $f : X \rightarrow K$  is a continuous map from a  $T_{3\frac{1}{2}}$  space to a compact  $T_2$  space, then there is a  $\beta f : \beta X \rightarrow K$  which makes the following diagram commute:*

$$\begin{array}{ccc} X & \xrightarrow{f} & K \\ h \searrow & & \nearrow \beta f \\ & \beta X & \end{array}$$

*Proof.* We prove this theorem by starting with a much version and building up.

1. Start with  $K = \mathbb{R}$  and  $f$  further required to be bounded. Then  $f$  is an  $f_\lambda$ . Define  $\beta f(h(x)) := h(x)(\lambda)$ .

2. Now let  $K$  be a product of closed bounded intervals. Let  $\beta f^\iota$  be the function from  $X$  to  $I_\iota$  satisfying  $f^\iota(x) = f(x)(\iota)$ . Define  $\beta f^\iota$  as in (1). Then, define  $\beta f(y) : \iota \mapsto \beta f^\iota(y)$ . [Check continuity use a sub-basic open set].
3. Now for full generality.  $K$  is  $T_{3\frac{1}{2}}$ , so it can be embedded in a product of closed bounded intervals,  $Z$ , say, as in the construction of the Stone-Čech compactification. Then, as in (2), we can find a  $\beta f : \beta X \rightarrow Z$  that makes this commute:

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ & \searrow h & \nearrow \beta f \\ & \beta X & \end{array}$$

But,  $\beta f : h(X) \rightarrow K$ , so  $\beta f : \overline{h(X)} \rightarrow \overline{K}$  and  $\overline{h(X)} = \beta X$  and  $\overline{K} = K$  as  $K$  is compact.

□

**Theorem 3.6 (Uniqueness).** *There is (up to homeomorphism) only one compactification with the Stone-Čech property. (And all compactifications with that property are equivalent in this sense of the ordering above).*

*Proof.* Suppose  $\langle k, \gamma X \rangle$  is another compactification of  $X$  with the Stone-Čech property. Define  $\beta k : \gamma X \rightarrow \beta X$  and  $\gamma h : \beta X \rightarrow \gamma X$  using the Stone-Čech properties. Then this sets up the equivalence mentioned above. □

**Theorem 3.7 (Maximality).**  *$\beta X$  is the greatest  $T_2$  compactification of a  $T_{3\frac{1}{2}}$  space,  $X$ .*

*Proof.* Use the Stone-Čech property. □

## 4 Metrisation.

Lectures: 9-11; Willard: pp. 165/6.

A space,  $\langle X, T \rangle$  is said to be *metrisable* iff there is some metric,  $d$ , which induces it.

**Definition 4.1.** *A space is Lindelöf iff every open cover has a countable subcover. A space is countably compact iff every countable open cover has a finite subcover.*

In metric spaces being Lindelöf is equivalent to being separable.

**Proposition 4.2.** *If  $X$  is Lindelöf  $T_3$ , then it is normal and hence  $T_4$ .*

*Proof.* Let  $F, G$  be disjoint, closed. For each  $y \in G$ , let  $A_y, V_y$  be disjoint open sets with  $C \subseteq A_y, y \in V_y$ .  $\{V_y : y \in G\}$  is an open cover of  $G$ , so we can find a countable subcover,  $\{V_{y_i} : i \in \mathbb{N}\}$ .

Do the same thing now for  $F$ : we end up with  $\{U_{x_i} : i \in \mathbb{N}\}$  a countable cover of  $F$ .

Define, by simultaneous recursion

$$\begin{aligned} S_1 &= U_{x_1} \\ T_1 &= V_{y_1} \setminus \overline{S_1} \\ S_{i+1} &= U_{x_{i+1}} \setminus (\overline{T_1} \cup \overline{T_2} \cup \dots \cup \overline{T_i}) \\ T_i &= V_{y_i} \setminus (\overline{S_1} \cup \dots \cup \overline{S_i}) \end{aligned}$$

Now, as  $F \subseteq A_y$ , which is open, and  $A_y \cap V_y = \emptyset$ , so  $\overline{V_y} \cap F = \emptyset$ . And similarly,  $\overline{U_x} \cap G = \emptyset$

This is the base step in an induction which will allow us to prove

$$\begin{aligned} S_{i+1} \cap F &= U_{x_{i+1}} \cap C \\ T_i \cap G &= V_{y_i} \cap D \\ \overline{S_{i+1}} \cap F &= \emptyset \\ \overline{T_i} \cap G &= \emptyset \end{aligned}$$

Let  $U := \bigcup_{i \in \mathbb{N}} S_i$ ,  $V := \bigcup_{i \in \mathbb{N}} T_i$ . These are the disjoint open sets separating  $F$  and  $G$ .  $\square$

A space is *first countable* iff each point has a countable neighbourhood base. A space is *second countable* iff its topology has a countable base.

**Proposition 4.3.** *If  $X$  is second countable, then it is Lindelöf.*

*Proof.* If  $\{B_n : n \in \mathbb{N}\}$  is a countable basis and  $\{U_\iota : \iota \in I\}$  is an open cover, for each  $n$ , define  $U_{\iota_n}$  such that  $B_n \subseteq U_{\iota_n}$  if one exists. Then  $\{U_{\iota_n} : n \in \mathbb{N}, U_{\iota_n} \text{ defined}\}$  is a countable subcover.  $\square$

**Theorem 4.4 (Urysohn's Metrisation Theorem).**  *$X$  is separable metrisable iff it is  $T_3$  and second countable.*

*Proof.* The only if is easy, we do the if. Let  $\mathcal{B}$  be a countable basis, then  $E = \{\langle B, B' \rangle : \overline{B} \subseteq B'\}$  is countable, say,  $E = \langle B_n, B'_n \rangle : n \in \mathbb{N}$ .

We will embed  $f$  homeomorphically in  $\prod_{n \in \mathbb{N}} [0, 1]$ , which is easy to metrise.

$X$  is  $T_4$  (as it's Lindelöf  $T_3$ ), so by Urysohn's Lemma,  $\overline{B_n}$  and  $X \setminus B'_n$  can be functionally separated by a continuous function  $f_n : X \rightarrow [0, 1]$ .

Define  $\phi : X \rightarrow \prod_{n \in \mathbb{N}} [0, 1]$  by  $\phi(x)(n) = f_n(x)$ .

It's quite a bit of not too challenging work to confirm this works.  $\square$

If  $Q, R \subseteq \mathcal{P}(X)$  then we say  $Q$  is a *refinement* of  $R$  iff for every  $A \in Q$  there is a  $B \in R$  such that  $A \subseteq B$ . We say  $Q$  is *locally finite* iff each  $x \in X$  has a neighbourhood which intersects only finitely many elements of  $Q$ .  $X$  is said to be *paracompact* iff every open cover of  $X$  has a locally finite open refinement.

**Theorem 4.5 (Stone's Theorem).** *Every metric space is paracompact.*

## 5 Connectedness.

Lectures: 8,9; PS: 4; Willard: pp. 193-201.

A space is *connected* iff it has no proper non-empty clopen subsets. It is *locally connected* iff for all  $x \in X$  and all open  $U \ni x$  there is a connected  $C$  and an open  $V$  such that  $x \in V \subseteq C \subseteq U$ .

Given a topological space  $X$ , define an equivalence relation  $\sim$  by  $x \sim y$  iff there is a connected  $C$  such that  $\{x, y\} \subseteq C$ . The equivalence classes under  $\sim$  are called *components*. Components are closed.

Given a point  $x \in X$ , the *quasi-component* of  $x$  is the intersection of all clopen sets containing  $x$ .

**Proposition 5.1.** *The quasi-components on a space,  $X$ , form a closed partition of  $X$  which is no finer than the component-partition.*

*Proof.* Quasi-components are closed as they are intersections of closed sets.

As for partitioning, what we must show is that if  $Q_x \neq Q_y$  then  $Q_x \cap Q_y = \emptyset$ . Suppose  $z \in Q_x \setminus Q_y$ . As  $z \notin Q_y$ , there's a clopen  $C \ni z, y \notin C$ . As  $z \in Q_x, x \in C$ . So,  $Q_x \subseteq C, Q_y \subseteq X \setminus C$ , so they're disjoint.

We now show that the component of  $x$  is contained in the quasi-component of  $x$ . Suppose  $y \notin Q_x$ . Let  $C$  be a clopen set such that  $y \notin C \ni x$ . If  $K \supseteq \{x, y\}$  then  $\{C \cap K, K \setminus C\}$  is a disconnection of  $K$ , so  $K$  is not connected. Hence,  $y$  is not in the component of  $x$ .  $\square$

**Theorem 5.2 (Sura-Bura Lemma).** *If  $X$  is compact  $T_2$ , then the components are quasi-components.*

*Proof.* We show that all quasi-components are connected. Suppose  $D$  is a quasi-component which has a non-empty proper clopen subset,  $E$ . Let  $F = D \setminus E$ .

As  $D$  is closed,  $E, F$  are closed in  $X$  and  $X$  is  $T_4$  (as it's compact  $T_2$ ), so there are disjoint open,  $U, V$  such that  $E \subseteq U, F \subseteq V$ .

Then,  $\{U, V\} \cup \{X \setminus C : C \text{ is clopen and } \supseteq D\}$  is an open cover, so has a finite subcover,  $\{U, V\} \cup \{X \setminus C_i : i = 1, \dots, n\}$ . So,  $\{U, V, \bigcup_{i=1}^n X \setminus C_i\}$  is also a cover. Write  $C := \bigcap_{i=1}^n C_i$ , then the cover can be written  $\{U, V, X \setminus C\}$ .

So,  $U \cup V \cup (X \setminus C) = X$ . Hence,  $C \subseteq U \cup V$ .  $C \cap U$  and  $C \cap V$  are disjoint, open and their union is  $C$ , so they are clopen in  $C$  and hence in  $X$  (as  $X$  is clopen). But, both have non-empty intersection with  $D$ , so  $D$  cannot be a quasi-component.  $\square$

**Theorem 5.3.** *A product of non-empty sets is connected iff each factor is.*