Revision Notes: Model Theory. (c1/2)

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FHS Part (II)

Notation:

Lectures are Prof. Zilber's 2003 series. I refer to his printed lecture notes which are available at http://www.maths.ox.ac.uk/ zilber/lect.ps

1 Structures.

Lectures: 1-6; Notes: pp. 1-11.

We define a language in the usual way: defining terms and formulae by recursion. We define a *structure* for a language, L, with non-logical symbols predicates P_i, functions f_i and constants c_i, to be a collection, $\mathcal{A} = \langle A; P_i^{\mathcal{A}}, f_i^{\mathcal{A}}, c_i^{\mathcal{A}} \rangle$, consisting of an underlying set (or *universe* or *domain*), subsets of $A^{<\omega}$, functions from $A^{<\omega}$ to A and members of A. We define truth and models by recursion in the usual way.

We may simplify arguments by treating functions and constants as predicates (say, let $P_f(x_0, \dots, x_n)$ hold iff $x_0 = f(x_1, \dots, x_n)$ and let $P_c(x)$ hold iff $x = c$).

Given two L-structures, $\mathcal{A}$, $\mathcal{B}$ we say $\mathcal{A}$ is *embedded* in (*isomorphic* to) $\mathcal{B}$, written $\mathcal{A} \subseteq \mathcal{B}$ ($\mathcal{A} \cong \mathcal{B}$), iff there is a map $\pi : A \to B$ which is injective (bijective) and respects predicates, functions and constants, which is to say, for predicates for example, $\mathbf{a} \in P_i^{\mathcal{A}} \Leftrightarrow \pi(\mathbf{a}) \in P_i^{\mathcal{B}}$.

Proposition 1.1. *Embeddings preserve atomic formulae.*

Proof. First show that they preserve all terms by induction on the complexity of terms. Then it pretty much follows by definition. □

Proposition 1.2. *Isomorphisms preserve all formulae.*

Proof. By induction on complexity of the formula. Slightly easier to use $\exists$ than $\forall$ as the quantifier. □

Corollary 1.3. *Let R be a relation on A. Then R is not L-definable if there is an automorphism (and isomorphism from $\mathcal{A}$ to $\mathcal{A}$) which doesn't fix R.*

A set, Σ, of L-sentence is *full* if for any sentence in Σ of the form $\exists v \psi(v)$ there is a closed L-term, λ, such that $\psi(\lambda) \in \Sigma$. It is *satisfiable* if it has a model. It is *finitely satisfiable* if every finite subset has a model.

Theorem 1.4 (Compactness theorem). *Any finitely satisfiable set of sentences is satisfiable.*

A structure is *canonical* iff for every $a \in A$ there is a closed L-term, λ, such that $a = \lambda^{\mathcal{A}}$.

The compactness theorem is a useful way to construct non-canonical models, eg. non standard models of arithmetic or analysis.

2 Diagrams and axiomatisations.

Lectures: 6-8; Notes: pp.12-8; Problem sheet 4.

We say an embedding, $\pi : \mathcal{A} \to \mathcal{B}$, is *elementary* if it preserves all L-formulae. In that case, we write, $\mathcal{A} \preceq \mathcal{B}$

Proposition 2.1 (Tarski's Lemma). *Suppose $\mathcal{A} \subseteq \mathcal{B}$ and (WLOG) $A \subseteq B$. Then $\mathcal{A} \preceq \mathcal{B}$ iff for all formulae, $\psi(x_1, \ldots, x_n)$, and all $a_1, \ldots, a_{n-1} \in A$ if there's a $b \in B$ such that $\mathcal{B} \models \psi(\mathbf{a}, b)$ then there's an $a' \in A$ such that $\mathcal{B} \models \psi(\mathbf{a}, a')$.*

Proof. First, note that the existence of such a b is equivalent to $\mathcal{B} \models \exists v \psi(\mathbf{a}, v)$.

If $\mathcal{A} \preceq \mathcal{B}$, then $\mathcal{B} \models \exists v \psi(\mathbf{a}, v)$ is equivalent to $\mathcal{A} \models \exists v \psi(\mathbf{a}, v)$ which is equivalent to the existence of an $a' \in A$ such that $\mathcal{A} \models \psi(\mathbf{a}, a')$. But, as $\mathcal{A} \preceq \mathcal{B}$, this is equivalent to $\mathcal{B} \models \psi(\mathbf{a}, a')$.

For the other direction, show that $\mathcal{A} \models \phi \Leftrightarrow \mathcal{B} \models \phi$ by induction on the complexity of ϕ. Atomic case is done using the embedding. Quantifier ($\exists$) case using the assumption that if $\mathcal{B} \models \exists v \psi(\mathbf{a}, v)$ then there's an $a' \in A$ for which $\mathcal{B} \models \psi(\mathbf{a}, a')$. $\square$

Given an L-structure, $\mathcal{A}$, we define $\mathcal{A}^+$ to be the natural extension which arises from enlarging L to $L_{\mathcal{A}}$, which contains a constant symbol, c_a, for each $a \in A$. Then, we define the *complete diagram*, $\mathrm{CDiag}\,\mathcal{A} := \{\sigma : \sigma \text{ is an } L_{\mathcal{A}}\text{-sentence and } \mathcal{A}^+ \models \sigma\}$. We define the *diagram* of $\mathcal{A}$, $\mathrm{Diag}\,\mathcal{A} := \{\sigma : \sigma \in \mathrm{CDiag}\,\mathcal{A} \text{ and } \sigma \text{ is atomic or the negation of an atomic formula}\}$.

Theorem 2.2 (Method of diagrams). *Given two L-structures, $\mathcal{A}$, $\mathcal{B}$, where L contains a constant symbol for each point in A, then $\mathcal{A} \subseteq \mathcal{B}$ iff $\mathcal{B} \models \mathrm{Diag}\,\mathcal{A}$. $\mathcal{B} \preceq \mathcal{B}$ iff $\mathcal{B} \models \mathrm{CDiag}\,\mathcal{A}$.*

A commonly used corollary of this is that if T is a set of L-sentences, then $T \cup \mathrm{Diag}\,\mathcal{A}$ ($T \cup \mathrm{CDiag}\,\mathcal{A}$) is satisfiable iff T has a model, $\mathcal{B}$ with $A \subseteq B$ ($A \preceq B$).

Theorem 2.3 (Upward Löwenheim-Skolem Theorem). *For any L-structure, $\mathcal{A}$, and cardinal, $\kappa \geq \max\{|L|, |A|\}$, there is an L-structure, $\mathcal{B}$, of size κ such that $\mathcal{A} \preceq \mathcal{B}$.*

Proof. Enlarge L to $(L_{\mathcal{A}})_\kappa$, a language including separate constant symbols for each member of A and a c_α for each $\alpha < \kappa$. Let $\Sigma := \mathrm{CDiag}\,\mathcal{A} \cup \{\neg c_\alpha = c_\beta : \alpha < \beta < \kappa\}$. Clearly, Σ is finitely satisfiable, so , by compactness, has a model $\mathcal{B}^+$. Let $\mathcal{B}$ be the the corresponding model for the language L. By the method of diagrams, this is what we wanted. $\square$

Theorem 2.4 (Downward Löwenheim-Skolem Theorem). *Let $\mathcal{B}$ be a non-empty L-structure, $S \subset B$, then there's an $\mathcal{A} \preceq \mathcal{B}$ with $S \subseteq A$ and $|A| \leq \max\{|S|, |L|\}$.*

Proof. We prove this using Skolem functions. Fix some $b_0 \in B$. For each L-formula, $\phi(x_1, \ldots, x_n)$ define the *Skolem function* of ϕ, $g_\phi : B^{n-1} \to B$ as follows:

$$g_\phi(b_1, \ldots, b_{n-1}) := \begin{cases} b & \text{if } b \in B \text{ is some element such that } \mathcal{B} \models \phi(b_1, \ldots, b_{n-1}) \\ b_0 & \text{if no such } b \text{ exists.} \end{cases}$$

Let A be the countable closure of S under the g_ϕs. That is: if $S_0 := S$; $S_{i+1} := \{g_\phi s_1, \ldots, s_n : s_1, \ldots s_n \in S_i, \phi \text{ an } L\text{-formula}\}$; then $A = \bigcup_{i \in \mathbb{N}} S_i$.

Endow A with the obvious predicates, functions and constants restricted from $\mathcal{B}$ to get a structure $\mathcal{A}$.

$\mathcal{A} \subseteq \mathcal{B}$ and if $\mathcal{B} \models \exists v \phi(\mathbf{a}, v)$ then $\mathcal{B} \models \phi(\mathbf{a}, g_\phi(\mathbf{a}))$, so the Tarksi lemma may be applied. $\square$

A class C of L-structures is *(finitely) axiomatisable* iff there is a (finite) set of sentences Σ, such that $\mathcal{A} \in C$ iff $A \models \Sigma$. It is $\exists$-axiomatisable ($\forall$-axiomatisable) iff there is such a set of sentences each of the form $\exists v_1 \cdots \exists v_n \phi$ ($\forall v_1 \cdots \exists v_n \phi$), where ϕ contains no quantifiers.

Proposition 2.5. *Let C be a class of structures which contains no infinite structures, but arbitrarily large finite structures. Then C is not axiomatisable.*

Proof. Let Σ be an axiomatisation. Write δ_n for the statement $\exists v_1 \cdots \exists v_n \bigwedge_{1<i<j<n} \neg v_i = v_j$.

Consider $\Sigma \cup \{d_i : i \in \mathbb{N}\}$. As C has arbitrarily large finite models, this is finitely satisfiable. Hence, by compactness, this has a model. But that model must be infinite. $\square$

Proposition 2.6. *Suppose Σ is an axiomatisation of a non-empty class, C. Then if C is finitely axiomatisable, it is axiomatised by a finite subset of Σ.*

Proof. Suppose T finitely axiomatises Σ, but no finite subset of Σ does. Hence, for each finite $\Sigma_0 \subset \Sigma$, there must be a structure $\mathcal{A} \models \Sigma_0$ with $\mathcal{A} \notin C$.

Write $T = \{\tau_1, \ldots, \tau_n\}$, $t := \neg \bigwedge_{i=1}^{n} \tau_i$.

Consider $\Sigma \cup \{t\}$. This is finitely satisfiable (as for each finite $\Sigma_0 \subset \Sigma$, we can find a model which isn't in C so satisfies t), so, by compactness, has a model, $\mathcal{B}$. But, $\mathcal{B} \in C$ and $\mathcal{B} \notin C$. $\square$

$\mathrm{Th}(C) := \{\sigma :$ *for all* $\mathcal{A} \in C$, $\mathcal{A} \models \sigma\}$. $\mathrm{Th}_\exists$ *and* $\mathrm{Th}_\forall$ *are the subsets consisting of the existential and absolute formulae.*

Proposition 2.7. *If C is axiomatisable then C is $\forall$-axiomatisable ($\exists$-axiomatisable) iff if $\mathcal{B} \in C$ and $\mathcal{A} \subseteq \mathcal{B}$ ($\mathcal{B} \subseteq \mathcal{A}$) then $\mathcal{A} \in C$.*

Proof. We do the only if of the $\forall$ version.

Suppose $\mathcal{A} \models \mathrm{Th}_\forall(C)$. Then $\mathrm{Th}(C) \cup \mathrm{Th}_\exists(\mathcal{A})$ is finitely satisfiable (use the fact that negations of existential formulae are absolute), so satisfiable. But this is equivalent to $\mathrm{Th}(C) \cup \mathrm{Diag}(\mathcal{A})$. Let $\mathcal{B}^+$ be some model of $\mathrm{Th}(C) \cup \mathrm{Diag}\,\mathcal{A}$ and $\mathcal{B}$ its equivalent over the original language. Then $\mathcal{B} \in C$ and $\mathcal{A} \subseteq \mathcal{B}$ (by method of diagrams). Hence, $\mathcal{A} \in C$. $\square$

3 Categoricity.

Lectures: 9-12; Notes: 19-28.

We henceforth use the word theory *to mean a deductively closed satisfiable set of sentences. For κ, a cardinal, we say a theory is κ-categorical iff it has only one (up to isomorphism) model of size κ.*

For K a field, the theory of K-vector spaces in the language L_K which has non-logical symbols $\{+, 0, \mu_k\}_{k \in K}$ is λ-categorical in $\lambda > |K|$. This is because the structure of an infinite vector space can be described completely in terms of the size of its basis and we have the result $|V| = |K| + \dim V$, if any of these are infinite.

A similar result may be proved for algebraically closed fields of given characteristic (for $\kappa > \aleph_0$) using transcendence degree rather than dimension. A transcendence basis is a maximal algebraically independent subset.

By a theorem of Cantor from b1, any two countable models of DLO (dense linear orders without end-points) are isomorphic, so DLO is $\aleph_0$-categorical.

Our aim for the rest of this section is to characterise $\aleph_0$-categoricity in terms of the number of equivalence classes as we now describe. Given a countable language, L, define

F_n to be the set of all L-formulae with n free variables. Given an L-theory, T, define an equivalence relation, E_n by $\langle \phi, \psi \rangle \in E_n$ iff $T \models \forall \mathbf{v}(\phi(\mathbf{v}) \leftrightarrow \psi(\mathbf{v}))$. F_n/E_n is called a Lindenbaum algebra. *It corresponds in an obvious way to definable subsets.*

*Proving theories with large Lindenbaum algebras non-$\aleph_0$-categorical will rely on looking at how they treat types, a gadget which we now define. A subset $p \subset F_n$ is an n-*type *iff p is closed under conjunctions and if $\phi \in p$ then $T \models \exists \mathbf{v} \phi(\mathbf{v})$. p is* complete *if for every $\phi \in F_n$, $\phi \in p$ or $\neg\phi \in p$. Given $\mathbf{a} \in A^n$,* $\mathrm{tp}(\mathbf{a}) := \{\phi \in F_n : \mathcal{A} \models \phi(\mathbf{a})\}$. *We say p is* realised *in $\mathcal{A}$ iff there is some $\mathbf{a} \in A^n$ such that $p \subseteq \mathrm{tp}(\mathbf{a})$. Otherwise, we say p is* omitted*.*

Lemma 3.1. *Given a countable model, $\mathcal{A}$, of T and an n-type p, we can find a $\mathcal{B} \succeq \mathcal{A}$ which realises p.*

Proof. Enlarge the language to L^+ by adding constant symbols, $c_1, \ldots, c_n$.

Consider $\mathrm{CDiag}(\mathcal{A}) \cup \{\phi(\mathbf{c}) : \phi \in p\}$. This is finitely satisfiable as any finite subset may only contain finitely many formulae from the type. As types are closed under conjunction, WLOG, there's just one. As any element of a type (when existentially bounded) is true we can find an allocation of the constants to elements of A which works. So, it has a model, $\mathcal{B}^+$, which we may take countable. By the method of diagrams, $\mathcal{A}$ elementarily embeds into it.

Let $\mathcal{B}$ be the L-reduct. p is still realised over this, by the tuple of elements we assigned to the constants in $\mathcal{B}^+$. ☐

A type, p, is said to be principal *if there is some $\phi \in F_n$ such that $T \models \exists \mathbf{v} \phi(\mathbf{v})$ and for all $\psi \in p$, $T \models \forall \mathbf{v}(\phi(\mathbf{v}) \to \psi(\mathbf{v}))$. Otherwise it is* non-principal*.*

Theorem 3.2 (Omitting Types Theorem). *If p is a non-principal n-type in a complete theory ,T, (over a countable language, L), then there is a countable model of T which omits p.*

We omit (tee hee) the proof of this theorem.

We say a formula ϕ is principal (for $\mathbf{a}$) *iff $\mathcal{A} \models \phi(\mathbf{a})$ and if $\psi(\mathbf{v})$ is such that $\mathcal{A} \models \psi(\mathbf{a})$ then $T \models \forall \mathbf{v}(\phi(\mathbf{v}) \to \psi(\mathbf{v}))$.*

Theorem 3.3 (Ryll-Nardzewski theorem). *T is $\aleph_0$-categorical iff F_n/E_n is finite for all $n \in \mathbb{N}$.*

Proof. We start with the only if. Suppose F_n/E_n is infinite. Let $p := \{\neg\phi_1 \wedge \cdots \wedge \neg\phi_k \in F_n : \phi_i$ are principal formulae$\}$. We show p to be an n-type.

Suppose not. p is clearly closed under conjunction, so we must have $T \models \forall \mathbf{v}(\phi_1(v) \vee \ldots \vee \phi_k(\mathbf{v}))$ for some principal formulae. Define $W_\psi := \{i \in \{1, \ldots, k\} : T \models \exists \mathbf{v}(\phi_i(\mathbf{v}) \wedge \psi(\mathbf{v}))\}$. Then, as the ϕ_i are principal, $\langle \psi, \chi \rangle \in E_n$ iff $W_\psi = W_\chi$. So, $|F_n/E_n| = 2^k$: contradiction.

So, p is non-principal, so by the omitting types theorem, there's a countable model which omits it. However, there is also another countable model which realises it: these models can't be isomorphic, so T is not $\aleph_0$-categorical.

Now for the if. Let $\mathcal{A}$, $\mathcal{B}$ be two models of T, and F_n/E_n finite. Suppose $A = \{a_1, \ldots a_n, \ldots\}$ and $B = \{b_1, \ldots, b_n, \ldots\}$. Note that the finiteness of F_n/E_n implies that every tuple has a principal formula (finite conjunctions of one from each equivalence class). We construct a new enumeration $a'_1, a'_2, \ldots,\ b'_1, b_2, \ldots$ to satisfy $\mathcal{A} \models \psi(a'_1, \ldots, a'_{n-1})$ iff $\mathcal{B} \models \psi(b'_1, \ldots, b'_{n-1})$ for all $\psi \in F_{n-1}$ at each stage. (This is true at the base stage as $\mathcal{A}$ and $\mathcal{B}$ are both models of T).

Suppose we've constructed up to a'_{n-1}, b'_{n-1}. Let a'_n be the first member of $\{a_1, a_2, \ldots\}$ not in $\{a'_1, \ldots, a'_{n-1}\}$. Let ϕ be a principal formula for $\langle a'_1, \ldots, a'_n \rangle$. Then, $\mathcal{A} \models \exists v \phi(a'_1, \ldots, a'_{n-1}, v)$, so (by inductive hypothesis), $\mathcal{B} \models \exists v \phi(b'_1, \ldots, b'_{n-1}, v)$, so there's some b'_n such that $\mathcal{B} \models \phi(b'_1, \ldots, b'_n)$. This satisfies the required condition, as if $\mathcal{A} \models \psi(a'_1, \ldots, a'_n)$ then, by principality of ϕ, $T \models \forall \mathbf{v}(\phi(\mathbf{v}) \rightarrow \psi(\mathbf{v}))$, so $\mathcal{B} \models \psi(b'_1, \ldots, b'_n)$.

We do the same the other way round at the next stage.

Once this is completed, we may define an isomorphism $a'_n \mapsto b'_n$.

□