

# Math 74 – Practice Questions for MT II.

Adam Booth.

Fall 2007.

The midterm will contain four questions – one on proofs involving algebra; one on proofs involving sets; one on relations; and one reading question. Each will have the same weight. Here’s some samples of each.

## 1 Algebra.

1. (This problem is based on Q22 on p. 123, which has a solution in the back of the book).

- (a) Explain how to prove an existential statement.  
(b) Why is the following proof wrong.

**Theorem 1.1.**  $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(xy^2 = y - x)$ .

*Proof.* Let  $x = \frac{y}{y^2+1}$ . Then,

$$\begin{aligned}y - x &= y - \frac{y}{y^2 + 1} \\ &= \frac{y^3}{y^2 + 1} \\ &= \frac{y}{y^2 + 1} \cdot y^2 \\ &= xy.\end{aligned}$$

□

- (c) Prove the ‘theorem’ is, in fact, false.

2. This question is based on Q 26 on p. 145, which has an answer in the back of the book.

- (a) How do you use a given which is a disjunction?  
(b) What’s wrong with the following theorem and proof?

**Theorem 1.2.** *For every real  $x$ , if  $|x - 3| < 3$ , then  $0 < x < 6$ .*

*Proof.* Let  $x$  be an arbitrary real and suppose  $|x - 3| < 3$ . We break into cases.  
*Case I:*  $x - 3 \geq 0$ . Then  $|x - 3| = x - 3$ . So, by assumption,  $x - 3 < 3$ , so we get that  $x < 6$ .

*Case II:*  $x - 3 < 0$ . Then  $|x - 3| = 3 - x$ . So, by assumption, we have that  $3 - x < 3$ . Solving for  $x$  gives that  $0 < x$ .

The two cases were exhaustive, so the theorem is proved. □

- (c) Write a correct proof of the theorem.

## 2 Set Theory.

1. (This problem is based on Q17 from p. 134, which has a solution in the back of the book).
  - (a) Explain how to prove that  $A \subseteq B$  for two sets  $A, B$ .
  - (b) Prove that  $\bigcup(\mathcal{F} \cap \mathcal{G}) \subseteq (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$ .
  - (c) What's wrong with the following proof that  $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$ ?

*Proof.* Suppose  $x \in (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$ . This means that  $x \in \bigcup \mathcal{F}$  and  $x \in \bigcup \mathcal{G}$ , so  $(\exists A \in \mathcal{F})(x \in A)$  and  $(\exists A \in \mathcal{G})(x \in A)$ . Thus we can choose a set  $A$  such that  $A \in \mathcal{F}$  and  $A \in \mathcal{G}$  and  $x \in A$ . Since  $A \in \mathcal{F}$  and  $A \in \mathcal{G}$ ,  $A \in \mathcal{F} \cap \mathcal{G}$ . Therefore,  $(\exists A \in \mathcal{F} \cap \mathcal{G})(x \in A)$ , so  $x \in \bigcup(\mathcal{F} \cap \mathcal{G})$ .  $\square$

2.
  - (a) Explain two ways of proving unique existence statements.
  - (b) Suppose  $\mathcal{F}$  is a set with the property that for every  $\mathcal{G} \subseteq \mathcal{F}$ ,  $\bigcup \mathcal{G} \in \mathcal{F}$ . Prove that there is a unique set  $A$  such that  $A \in \mathcal{F}$  and  $(\forall B \in \mathcal{F})(B \subseteq A)$ .
  - (c) If you drop the hypothesis on  $\mathcal{F}$ , is the result still true? If not, can you still prove existence, but not uniqueness; or can you still prove uniqueness, but not existence?

## 3 Relations.

1.
  - (a) Suppose  $R$  is a relation on  $X \times X$ . Define the following terms:
    - i. symmetric;
    - ii. reflexive;
    - iii. transitive.
  - (b) Now suppose that  $\mathcal{F} \subseteq \mathcal{P}(A)$  and  $A$  and  $B$  are sets. We define a relation  $F$  on  $Y \times Y$ , where  $Y = \{f : f \text{ is a function from } A \text{ to } B\}$ , by  $(f, g) \in F$  iff  $\{x \in A : f(x) = g(x)\} \in \mathcal{F}$ .
    - i. Prove that  $\mathcal{F}$  is symmetric.
    - ii. Give with proof a condition on  $\mathcal{F}$  which equivalent to  $R$  being reflexive.
    - iii. Prove that if  $\mathcal{F}$  is closed under superset<sup>1</sup> and intersection<sup>2</sup>, then  $\mathcal{F}$  is transitive.
2.
  - (a) Let  $R$  be a relation on  $A \times A$ . Define the following terms.
    - i. A partial order. [You may assume that reflexive, symmetric, anti-symmetric and transitive are already understood terms.]
    - ii. Smallest element.
    - iii. Minimal element.
  - (b) Let  $A$  be the set of finite strings of 0s and 1s (including the empty string) and define  $R$  to be the relationship  $xRy$  iff  $y$  is  $x$  with some extra digits stuck on the end (or  $y = x$ ).

---

<sup>1</sup>ie. if  $D \in \mathcal{F}$  and  $D \subset E \subset A$ , then  $E \in \mathcal{F}$

<sup>2</sup>ie. if  $D, E \in \mathcal{F}$  then  $D \cap E \in \mathcal{F}$

- i. Check that  $R$  is a partial ordering.
- ii. Is  $R$  a total ordering?
- iii. Does  $A$  have minimal elements? Smallest elements? [List them all and explain why.]
- iv. Find a subset of  $A$  which, regarded as an ordered set in its own right, has minimal elements but no smallest element.