Math 74 – Practice Questions for the final.

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The final will contain 7 questions: one on orderings; one on equivalence relations; one on functions; two on induction; one on equinumerosity; and one reading question. Here are some samples.

1 Orderings.

1. This question is loosely based on questions 6, 8 and 9 from section 4.4. 6 has a solution in the homework solutions and 8 has a solution in the back of the book.
   (a) Define what it means for a relation R to be a partial ordering and a total ordering.
   (b) Let R be a total ordering on A and S a total ordering on B. Define a relations T_1, T_2, T_3 on $A \times B$ as follows:

$$(a,c)T_1(b,d) :\equiv aRb \wedge cSd$$
$$(a,c)T_2(b,d) :\equiv aRb \vee cSd$$
$$(a,c)T_3(b,d) :\equiv aRb \vee (a = b \wedge cSd)$$

   Must T_1, T_2, T_3 be partial orders? Must they be total? Justify your answers with proofs or counter-examples. [This question is longer than anything I'd put on the real final, but all the bits of suitable.]

2. (a) Define what minimal and smallest elements of partial orders are.
   (b) Show that if a set A has a smallest element with respect to some partial order R, then it has exactly one.
   (c) Is the same true for a pre-order? [See Q1 on Equivalence relations for the definition and a hint.]

2 Equivalence relations.

1. (a) Define what it means for a relation to be an equivalence relation.
   (b) Recall that a *pre-ordering* is a relation which satisfies all of the properties of a partial order except anti-symmetry. Prove that if R is a pre-ordering, then $R \cap R^{-1}$ is an equivalence relation.

(c) Let R be the following relation on the set of (real) $n \times n$ matrices: ARB iff there is a matrix C such that $B = AC$. You may assume R is a pre-order, and hence that $S = R \cap R^{-1}$ is an equivalence relation. What is the equivalence class of the identity matrix?

2. (a) Define what it means for a relation to be an equivalence relation.
   (b) Show, by giving an example and explaining how it fails, that the relation on $\mathbb{R}$, xRy iff $|f(x) - f(y)| < 1$, need not be an equivalence relation, where f is some given function from $\mathbb{R}$ to $\mathbb{R}$.
   (c) Give two examples of functions f that do give equivalence relations when used as in (b). One should infinitely have many equivalence classes and the other should have finitely many. (And you should say which is which!)

3 Functions.

1. (a) If $f : X \to Y$, $A \subseteq X$ and $B \subseteq Y$ then define $f[A]$ and $f^{-1}(B)$.
   (b) Prove or give a counterexample: $f^{-1}(f[A]) = A$.
   (c) Prove or give a counterexample: $f[f^{-1}(B)] = B$.
2. (a) Define the terms *onto*, *1:1* and *bijection*.
   (b) A function $f : A \to A$ is said to be *right cancelable* if for all functions $g, h : A \to A$, if $g \circ f = h \circ f$ then $g = h$. Prove or disprove the following claims:
      i. f is right-cancelable iff it is 1:1.
      ii. f is right-cancelable iff it is onto.

4 Induction.

Recall, there will be two induction questions on the final.

1. This question is based on questions 10 and 14 of 6.1, which have answers in the back of the book.
   (a) Prove by induction that for all $n \in \mathbb{N}$, $64 \mid (9^n - 8n - 1)$.
   (b) Prove by induction that for all $n \in \mathbb{N}$, if $n \geq 10$ then $2^n > n^3$.
2. (This question is based on question 6 from section 6.4 and has a solution in the back of the book.) Recall that the Fibbonacci numbers are defined as

$$\begin{aligned} F_0 &:= 1 \\ F_1 &:= 1 \\ F_n &:= F_{n-2} + F_{n-1} \qquad\qquad n \geq 2. \end{aligned}$$

   (a) Prove that for all n, $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.
   (b) Prove that for all n, $\sum_{i=0}^{n} F_{2i+1} = F_{2n+2} - 1$.
   (c) Hence, write down a formula for $\sum_{i=0}^{n} F_{2i}$.

3. A strict partial order[1] R on a set A is said to be *dense in itself* if for all $x, y \in A$, if xRy then there is a z such that $xRzRy$. Show, by induction, that no non-empty strict linear order on a finite set is dense in itself.

4. A sequence $a_0, a_1, a_2, \ldots$ is defined recursively as

$$\begin{aligned} a_0 &= 0 \\ a_{n+1} &= (a_n)^2 + \frac{1}{4}. \end{aligned}$$

Prove, by induction, that for all $n > 0$, $0 < a_n < 1$. [Hint: it is actually easier to prove that $0 < a_n < l$, where l is some number you should determine.] [This question is based on question 19 from 6.3 which has an answer in the back of the book.]

5 Infinite Sets.

1. (a) Define what it means for two sets to be *equinumerous* and state the Schröder-Bernstein theorem.
   (b) Use the Schröder-Bernstein theorem to show that $\mathbb{N}$ is equinumerous to the set $P = \{2^n 3^m : n, m \in \mathbb{N}\}$.
   (c) Hence, show that $\mathbb{N}$ and $\mathbb{Q}_{\geq 0}$ are equinumerous.

2. (a) Define what it means for two sets to be *equinumerous* and state the Schröder-Bernstein theorem.
   (b) In the rest of this question we will show that $\mathcal{P}(N)$ is equinumerous with $\mathbb{N}^{\mathbb{N}}$ (the set of functions from $\mathbb{N}$ to $\mathbb{N}$). You may assume that some bijection $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is given.
      i. Use π to build a 1:1 function $f : \mathbb{N}^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$.
      ii. Find a 1:1 function going the other way.

[1] A strict partial order is one which doesn't satisfy reflexivity and satisfies a stronger form of antisymmetry: $(\forall a, b)\neg((a, b) \in R \wedge (b, a) \in R)$.