53 Quiz Solutions

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Math 53 – Quiz 1 – 7/5/06.

1. Answer **exactly two** out of the following three questions in complete sentences, making sure your notation is clear.

   (a) How do you find the slope of a tangent line to a polar curve?
   If a polar curve has equation $r = f(\theta)$, then the tangent to the point $(r(\theta_0), \theta_0)$ is

   $$\frac{f'(\theta_0)\sin\theta_0 + f(\theta_0)\cos\theta_0}{f'(\theta)f'(\theta_0)\cos\theta_0 - f(\theta_0)\sin\theta}.$$

   (b) How do you find the area of a region bounded by a polar curve?
   The area bounded by the lines $\theta = \alpha$, $\theta = \beta$ and the arc of the curve $r = f(\theta)$ between α and β is $\int_\alpha^\beta \frac{1}{2}[f(\theta)]^2\, d\theta$.

   (c) How do you find the length of a polar curve?
   The length of the arc of the curve $r = f(\theta)$ between $\theta = \alpha$ and $\theta = \beta$ is $\int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$.

2. Identify the type of conic section whose equation is given and sketch it, labelling the vertices and foci:

   $$x^2 = 4y - 2y^2.$$

   Completing the square gives us

   $$\frac{1}{2}x^2 + 1(y-1)^2 = 1$$

   which is an ellipse with vertices at $(\pm\sqrt{2}, 1)$ and foci at $(\pm 1, 1)$.

3. Two particles move according to the following equations:

   $$x_1 = a + b\sin(t) \qquad x_2 = \cos(t)$$
   $$y_1 = c + d\cos(t) \qquad y_2 = \sin(t)$$

   where a, b, c, d are all constants. By drawing pictures, show that the two paths may have 0, 1, 2, 3, 4 or infinitely many points of intersection. For each picture, give values of a, b, c, d which produces something like what you've drawn.

For 0, we can just take some circle which gets nowhere near the second shape, say $(a, b, c, d) = (10, 1, 0, 1)$.

To get 1, we need something which just touches the unit circle, like $(2, 1, 0, 1)$.

To get 2, we need something which passes through the circle, like $(1, 1, 0, 1)$.

To get 3, it needs to pass through the circle on one side and just touch it on the other, like $(-1, 2, 0, 1)$.

To get 4, it needs to pass through boths sides, like $(0, 2, 0, \frac{1}{2})$.

To get infinitely many, it has to be the same shape, ie. we need $(0, 1, 0, 1)$.

Math 53 – Quiz 2 – 7/7/06.

1. Write expressions for the scalar and vector projections of $\mathbf{b}$ onto $\mathbf{a}$. Illustrate with diagrams.

   The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

   $$\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$$

   and the vector projection is the scalar projection multiplied by

   $$\frac{\mathbf{a}}{|\mathbf{a}|}.$$

2. If $\mathbf{a} = \langle 1, 2 \rangle$ and $\mathbf{b} = \langle -2, 1 \rangle$, evaluate $\mathbf{a} + 3\mathbf{b}$, $|\mathbf{a}|$ and $\mathbf{a} \cdot \mathbf{b}$. What's the angle between $\mathbf{a}$ and $\mathbf{b}$?

   $$\begin{aligned} \mathbf{a} + 3\mathbf{b} =& \langle 1, 2 \rangle + \langle -6, 2 \rangle \\ =& \langle -5, 4 \rangle \\ |\mathbf{a}| =& \sqrt{1 + 2^2} \\ =& \sqrt{5} \\ \mathbf{a} \cdot \mathbf{b} =& -2 + 2 \\ =& 0. \end{aligned}$$

   The angle between $\mathbf{a}$ and $\mathbf{b}$ is a right-angle.

3. Write an inequality to describe the set of points which are closer to the origin than to $(1, 1, 1)$. Sketch the region of the (xy)-plane that is included. [Hint: you might want to simplify the inequality after you write it down.]

If (x, y, z) is closer to $(0, 0, 0)$ than it is to $(1, 1, 1)$, then it satisfies

$$x^2 + y^2 + z^2 < (x - 1)^2 + (y - 1)^2 + (z - 1)^2$$

which simplifies (by expanding brackets and cancelling squares) to

$$2x + 2y + 2z < 3$$

Looking just at the (xy)-plane corresponds to setting $z = 0$, which means the the region we want is the one below the line $y = \frac{3}{2} - x$.

Math 53 – Quiz 3 – 7/11/06.

1. (a) Write the equations for converting *from* cylindrical *to* rectangular coordinates. Draw a picture show what r, θ and z represent.

   The equations are:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

   (b) Do the same for spherical coordinates.

$$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}$$

2. Find an equation for the plane passing through the points $(1,2,1)$, $(-2,1,0)$ and $(3,0,0)$

   To find the equation of a plane we need two things: a point on it (check) and a vector normal to it. We don't have the latter yet, so let's find one by first finding two vectors in the plane. Name the points given A, B, C. Then $\bar{AC} = \langle -2, 2, 1\rangle$ and $\bar{BC} = \langle -5, 1, 0\rangle$ are in it. Their cross-product will be a vector normal to it. This is:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 1 \\ -5 & 1 & 0 \end{vmatrix} = \langle -1, -5, 8\rangle.$$

   Hence, an equation of the plane is:

$$\langle -1, -5, 8\rangle \cdot (\mathbf{r} - \langle 3, 0, 0\rangle) = 0.$$

3. Suppose $\mathbf{a}$ and $\mathbf{b}$ are 3D-vectors and p and q are positive real numbers. Show that $\mathbf{a}\times\mathbf{b}$ is parallel to

$$(\mathbf{a} + p\mathbf{b}) \times (\mathbf{a} + q\mathbf{b}).$$

$$\begin{aligned} (\mathbf{a} + p\mathbf{b}) \times (\mathbf{a} + q\mathbf{b}) =& \underbrace{\mathbf{a}\times\mathbf{a}}_{=0} + q\mathbf{a}\times\mathbf{b} + p\underbrace{\mathbf{b}\times\mathbf{a}}_{=\mathbf{a}\times\mathbf{b}} + pq\underbrace{\mathbf{b}\times\mathbf{b}}_{=0} \\ =&(p-q)\mathbf{a}\times\mathbf{b} \end{aligned}$$

Math 53 – Quiz 4 – 7/14/06.

1. (a) What is a function of two variables?

   A function of two variables is a rule which gives a unique real number when fed a real number into each of two input slots.

   (b) Describe two methods for visualizing a function of two variables.

   One method is a contour plot: this consists of drawing curves of the form $f(x, y) = k$ for various values of k on an xy-plane. These curves are known as level curves (or, more generally, level *sets*.)

   Another way is to draw a 3D graph. This is usually done by drawing traces in the vertical planes $x = k$ and $y = k$ for equally spaced values of k. Parts of the graph are then eliminated using hidden line removal.

2. Find parametric equations for the tangent line to the curve with parametric equations

$$x = t^2 + t,\ y = \sin(t),\ z = e^t$$

   at the point $(0, 0, 1)$.

   To find equations for a line we need a point on it (check), and a vector along it. To find the latter, we must differentiate the vector function $\langle t^2 + t, \sin(t), e^t\rangle$ and evaluate that at $t = 0$ (as this is the only value of t which gives us the point $(0, 0, 1)$).

$$\frac{d}{dt}\langle t^2 + t, \sin(t), e^t\rangle = \langle 2t + 1, \cos(t), e^t\rangle$$
$$\therefore \frac{d}{dt}\langle t^2 + t, \sin(t), e^t\rangle \restriction_{t=0} = \langle 1, 1, 1\rangle$$

   So, the vector equation of the line is $\mathbf{r}(s) = \langle 0, 0, 1\rangle + s\langle 1, 1, 1\rangle$. Translating this into a parametric equation gives us

$$x = s,\ y = s,\ z = 1 + s.$$

3. Find the first derivatives of the following function:

$$u = x_1 + x_1x_2 + x_1x_2x_3 + \ldots + x_1x_2\cdots x_n.$$

This function is linear in x_k for any $k = 1, \ldots, n$. All the terms before the first appearance of x_k drop out, and the x_k disappears from all the other terms, leaving us with:

$$\begin{aligned}\frac{\partial}{\partial x_k}(x_1 + x_1x_2 + x_1x_2x_3 + \ldots + x_1x_2\cdots x_n) &= \frac{x_1x_2\cdots x_k + x_1x_2\cdots x_kx_{k+1} + \ldots + x_1x_2\cdots x_n}{x_k}\\ &= x_1x_2\cdots x_{k-1}(x_{k+1} + x_{k+1}x_{k+2} + \ldots + x_{k+1}x_{k+2}\cdots x_n).\end{aligned}$$

Math 53 – Quiz 5 – 7/18/06.

1. State the chain rule where $z = f(x, y)$ and x and y are functions of one variable. What if x and y are functions of two variables? [As usual, answer in full sentences and make sure your notation is clear.]

   If $x = x(t)$ and $y = y(t)$ then

   $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

   If, however, $x = x(s, t)$ and $y = y(s, t)$ then

   $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
   $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

2. Find the directional derivative of $f(x, y, z) = \frac{x}{y+z^2}$ at the point $(4, 2, 1)$ in the direction of the vector pointing from this point to the origin.

   The gradient of f is

   $$\nabla f = \langle \frac{1}{y+z^2}, \frac{-x}{(y+z^2)^2}, \frac{-2xz}{(y+z^2)^2} \rangle.$$

   , when we evaluate this at $(4, 2, 1)$, we get $\langle \frac{1}{3}, \frac{-4}{9}, \frac{-8}{9} \rangle$. The direction we want to find the derivative at is $\langle -4, -2, -1 \rangle$. The norm of this vector is $\sqrt{21}$, so the corresponding unit vector is $\langle \frac{-4}{\sqrt{21}}, \frac{-2}{\sqrt{21}}, \frac{-1}{\sqrt{21}} \rangle$. So, the directional derivative we're looking for is

   $$\langle \frac{1}{3}, \frac{-4}{9}, \frac{-8}{9} \rangle \cdot \langle \frac{-4}{\sqrt{21}}, \frac{-2}{\sqrt{21}}, \frac{-1}{\sqrt{21}} \rangle = \frac{-4}{9\sqrt{21}}.$$

3. Estimate the maximum possible error in a calculation of x^y where x and y are numbers between 100 and 200 rounded to the nearest integer.

Recall that $x^y = e^{y\ln(x)}$. Hence, if $z = x^y$ then,

$$\begin{aligned} dz &= dx(\frac{y}{x}e^{y\ln(x)}) + dy(\ln(x)e^{y\ln(x)}) \\ &= dx(yx^{y-1}) + dy(\ln(x)e^{y\ln(x)}). \end{aligned}$$

dx and dy around bounded by 0.5; the other functions in the formula increase as x, y increase so we put 200 in for x and y. So, dz is bounded by

$$(200^{200} + 200^{200}\log(200))/2.$$

Math 53 – Quiz 6 – 7/21/06.

1. (a) Give a clear statement of the Extreme Value Theorem for Functions of Two Variables.
   If a function, f, is continuous on a closed bounded subset, D, of $\mathbb{R}^2$ then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ for some points (x_1, y_1) and (x_2, y_2) in D.

   (b) Are the following subsets of $\mathbb{R}^2$ closed? Bounded? If you claim one is bounded, give a bound; if you claim one isn't closed, give a boundary point which isn't contained in the set.

   i. $S^1 := \{(x, y) : x^2 + y^2 = 1\}$.
   Bounded by 1, closed.

   ii. $\{(x, y) : 0 \le y < 1\}$.
   Not bounded; not closed, $(0, 1)$ is a boundary point which isn't in the set.

   iii. $\{(0, 0), (1, 3), (-4, 0)\}$.
   Bounded by 4; closed.

2. Find the local maximum and minimum values and saddle points of $f(x, y) = x^3y + 12x^2 - 8y$.

   Let's first find the derivatives:

$$
\begin{aligned}
f_x(x, y) &= 3x^2y + 24x & f_y(x, y) &= x^3 - 8 \\
f_{xx}(x, y) &= 6xy + 24 & f_{yy}(x, y) &= 0 \\
f_{xy} &= 6x.
\end{aligned}
$$

   At critical points, we have $f_y(x, y) = 0$, so $x = 2$. Hence, $y = \frac{(-24)(2)}{(3)(2)^2} = -4$. Plugging in the appropriate values, we get that $D = -144 < 0$, so this is a saddle point.

3. If the length of the diagonal of a rectangular box must be L, what is the largest possible volume?

   We have three variables: x, y, z. We are trying to maximise $V = xyz$. However, we also have the constraint $x^2 + y^2 + z^2 = L^2$ and $x, y, z \ge 0$. So, we pass to the two-variable problem of maximizing

$$V = xy\sqrt{L^2 - x^2 - y^2}$$

   subject to $x, y \ge 0$, $x^2 + y^2 \le L^2$, ie. we're maximising over a quarter-circle. We also note that if x, y or $z = 0$, then the volume is zero, so we don't have to bother checking along the curves.

   So, we just have the 2D optimization to do. Instead of working with V which has an annoying square root in, instead we'll maximize

$$W = V^2 = x^2y^2(L^2 - x^2 - y^2).$$

   This has the following first derivatives:

$$
\begin{aligned}
W_x &= 2xy^2(L^2 - x^2 - y^2) - 2x^3y^2 \\
&= 2xy^2L^2 - 2xy^4 - 4x^3y^2 \\
W_y &= 2yx^2L^2 - 2yx^4 - 4y^3x^2
\end{aligned}
$$

(The formula for W is symetric in x and y so there's no need to do any calculations for W_y). We set these equal to 0 and cancel xs, ys (as we can assume x and y aren't 0 as that would give us a volume of 0) and a 2, getting.

$$L^2 - 2x^2 - y^2 = 0$$
$$L^2 - 2y^2 - x^2 = 0.$$

Solving these equations simultaneously gives $x = y = \sqrt{L/3}$ (we may reject the negative square root, as we're requiring $x, y, z \geq 0$). Hence, the largest possible volume is $(L/3)^{3/2}$.

Math 53 – Quiz 7 – 7/25/06.

1. Suppose f is a continuous function defined on a rectangle $R = [a, b] \times [c, d]$.
   (a) Write the definition of $\iint_R f(x, y)\, dA$ as a limit.
   The integral is defined as the following limit:

$$\lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A$$

   where $x_i^* \in [a+(i-1)(b-a)/n, a+i(b-a)/n]$, $y_j^* \in [c+(j-1)(d-c)/n, c+j(d-c)/n]$ and $\Delta A := (b-a)(d-c)/(mn)$.

   (b) What is the geometric interpretation of $\iint_R f(x, y)\, dA$ if $f(x, y) \geq 0$?
   It is the volume of the solid bounded by the planes $z = 0$, $x = a$, $x = b$, $y = c$, $y = d$ and the surface $z = f(x, y)$.

   (c) How do you evaluate $\iint_R f(x, y)\, dA$?
   Using Fubini's theorem, which tells us that $\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy$.

2. Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = 4x + 6y$ subject to the constaint $x^2 + y^2 = 13$.

   We have to solve the equations:

$$\begin{aligned} 4 &= 2\lambda x & (I) \\ 6 &= 2\lambda y & (II) \\ x^2 + y^2 &= 13 & (III) \end{aligned}$$

   Solving (I) and (I) for x and y (noting that λ can't be 0 as then we'd have $4 = 2(0)x = 0$) and substituting into (III) gives

$$\frac{4}{\lambda^2} + \frac{9}{\lambda^2} = 13$$

   which has solutions ± 1. $\lambda = 1$ gives us the solution $(x, y) = (2, 3)$ and $f(2, 3) = 26$; $\lambda = -1$ gives us the solution $(x, y) = (-2, -3)$ and $f(-2, -3) = -26$. These are the maximum and minimum respectively.

3. Use Lagrange multipliers to prove that the triangle with maximum area that has perimeter 2 is equilateral. [You might like to use Heron's formula for the area: $A = \sqrt{(1-x)(1-y)(1-z)}$ where x, y, z are the lengths of the sides. (This formula is only valid when the perimeter is 2).]

So as not to have to deal with the square root, let's maximize

$$B = (1-x)(1-y)(1-z)$$

subject to

$$g(x, y, z) = x + y + z = 1.$$

Lagrange multipliers gives us the following equations to solve:

$$
\begin{aligned}
-(1-y)(1-z) &= \lambda && (I)\\
-(1-x)(1-z) &= \lambda && (II)\\
-(1-x)(1-y) &= \lambda && (III)\\
x+y+z &= 2 && (IV).
\end{aligned}
$$

Using (II) and (III), we can get

$$(1-x)(1-z) = (1-x)(1-y).$$

So long as $x \neq 1$, we can cancel the $(1-x)$. But, if $x = 1$ then the area is 0, by Heron's formula, so we can ignore this case and conclude that, for the maximum,

$$y = z$$

Substituting this into (I) gives

$$-(1-y)^2 = \lambda$$

and substituting this into (III) gives

$$(1-x)(1-y) = (1-y)^2$$

By the same argument as above for x, y can't be 1 at the maximum area, so we can cancel $(1-y)$s and simplify to get

$$x = y$$

Job done. (We've already got $y = z$, so we have $x = y = z$, so the triangle is equilateral. Note: we never used (IV), which shouldn't actually be too suprising).

Math 53 – Quiz 8 – 7/28/06.

1. If a lamina occupies a plane region, D, and has density function $\rho(x, y)$, write expressions for each of the following in terms of double integrals.

   (a) The mass.

$$\iint_D \rho(x, y)\, dA$$

   (b) The center of mass.

$$(\bar{x}, \bar{y}) = \frac{1}{\iint_D \rho(x, y)\, dA} \left(\iint_D x\rho(x, y)\, dA, \iint_D y\rho(x, y)\, dA \right)$$

   (c) Which of the above integrals could come out negative?
   The mass couldn't; the coordinates of the center of mass could.

2. Evaluate the following integral by reversing the order of integration:

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3 + 1}\, dx\, dy.$$

   Drawing a picture, we see that x-values between 0 and 1 are used and, given a value of x, y ranges between 0 and x^2. Hence, the integral we want to do is

$$\int_0^1 \int_0^{x^2} \sqrt{x^3 + 1}\, dy\, dx = \int_0^1 x^2 \sqrt{x^3 + 1}\, dx \qquad u = x^3 + 1$$

$$du 3x^2\, dx$$

$$= \frac{1}{3} \int_1^2 \sqrt{u}\, du$$
$$= \frac{2}{9} u^{3/2}]_1^2$$
$$= \frac{2}{9}(2^{3/2} - 1).$$

3. Aloysius and Betty are the only two clerks at a store. Each of them has their own line for their checkout. The wait-time in Aloysius' line is uniformally distributed between 1 and 7 minutes. The wait-time in Betty's line is distributed according to an exponential distribution with mean 4 minutes. The wait-time at each is independent of the wait-time at the other. If Anne joins Aloysius' line and Barak Betty's at the same time, what is the expected amount of time that whoever is dealt with first has to wait for the other?

The marginal p.d.f for Aloysius' line is

$$f_A(x) := \begin{cases} \frac{1}{6} & x \in [1, 7] \\ 0 & x \notin [1, 7]. \end{cases}$$

The marginal p.d.f. for Betty's line is

$$f_B(y) := \begin{cases} \frac{1}{4} e^{-y/4} & y \geq 0 \\ 0 & y < 0. \end{cases}$$

As they are independent, the j.d.f. is the product of the marginals:

$$f_{A,B}(x,y) := \begin{cases} \frac{1}{24}e^{-y/4} & (x,y) \in [1,7] \times [0,\infty) \\ 0 & (x,y) \notin [1,7] \times [0,\infty) \end{cases}$$

The expectation we're looking for is $E|X-Y| = \iint_{\mathbb{R}^2} |x-y| f_{A,B}(x,y)\, dA$. As usual, we deal with the absolute value function by breaking the domain into two parts: one on which $x > y$ and one on which $y < x$. We also deal with the cases in the definition of f by just integrating over the support of the function (that is, the locus of points on which the function isn't 0). Drawing a picture, we get that

$$\begin{aligned} E|X-Y| &= \int_1^7 \int_0^x \frac{x-y}{24} e^{-y/4}\, dy\, dx + \int_1^7 \int_x^\infty \frac{y-x}{24} e^{-y/4}\, dy\, dx. \\ &= \int_1^7 \int_0^x \frac{x}{24} e^{-y/4}\, dy\, dx - \int_1^7 \int_0^x \frac{y}{24} e^{-y/4}\, dy\, dx \\ &\quad + \int_1^7 \int_x^\infty \frac{y}{24} e^{-y/4}\, dy\, dx - \int_1^7 \int_x^\infty \frac{x}{24} e^{-y/4}\, dy\, dx \\ &= \int_1^7 \frac{x}{6}(1 - e^{-x/4})\, dx + \int_1^7 \frac{1}{6}(e^{-x/4}(x+4) - 4)\, dx \\ &\quad + \int_1^7 \frac{1}{6} e^{-x/4}(x+4)\, dx - \int_1^7 \frac{x}{6} e^{-x/4}\, dx \\ &= \frac{1}{6} \int_1^7 (x + 8e^{-x/4} - 4)\, dx \\ &= \frac{1}{6} \left[\frac{x^2}{2} - 32e^{-x/4} - 4x \right]_1^7 \\ &= \frac{16}{3}(e^{-1/4} - e^{-1/7}). \end{aligned}$$

Math 53 – Quiz 9 – 8/1/06.

1. (a) Write an expression for the area of a surface with equation $z = f(x, y)$, $(x, y) \in D$.

   The surface area is $\iint_D \sqrt{1 + z_x^2 + z_y^2} dA$.

   (b) Under what circumstances could the above expression come out negative? Zero?

   It can never be negative. It could only be zero if D wasn't really a region of the plane, but a curve or a point.

2. Evaluate $\iiint_E xz\, dV$ where E is the solid tetrahedron with vertices $(0, 0, 0), (0, 1, 0), (1, 1, 0), (0, 1, 1)$.

   Three of the faces are parallel to axes; the one that isn't is the one with vertices $(0, 0, 0), (1, 1, 0), (0, 0, 1)$. This face is a plane containing the point $(0, 0, 0)$ and the vectors $\langle 1, 1, 0\rangle$ and $\langle 0, 1, 1\rangle$. Hence, one of its normal vectors is $\langle 1, -1, 1\rangle$, so it has equation $x - y + z = 0$. The region of the xy-plane that the region sits on is the half of the square with corners $(0, 0)$ and $(1, 1)$ above the line $x = y$. We set up our integral with z on the inside, to take advantage of the way we've packaged up the information so far. The integral is:

$$\begin{aligned} \int_0^1 \int_x^1 \int_0^{y-x} xz\, dz\, dy\, dx &= \frac{1}{2} \int_0^1 \int_x^1 x(y - x)^2\, dy\, dx \\ &= \frac{1}{6} \int_0^1 x(1 - x)^3\, dx \qquad\qquad u = 1 - x \\ &= \frac{1}{6} \int_0^1 u^3 - u^4\, du \\ &= \frac{1}{120}. \end{aligned}$$

3. Rewrite the integral $\int_0^1 \int_y^1 \int_0^y f(x, y, z)\, dz\, dx\, dy$ as an integral with differentials in the order $dy\, dx\, dz$.

   Drawing a picture, we see that this region is a tetrahedron with vertices at $(0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1)$. So, z ranges from 0 to 1. Given a value of z, x runs from z to 1. Given a value of x and z which actually cuts through the tetrahedron, y runs from z to x. Hence the integral is

$$\int_0^1 \int_z^1 \int_z^x f(x, y, z)\, dy\, dx\, dz.$$

Math 53 – Quiz 10 – 8/4/06.

1. (a) How do you change from rectangular coordinates to cylindrical coordinates in a triple integral?

   You need to change to integrand (change x to $r\cos\theta$ and y to $r\sin\theta$), the differential (change $dx\,dy\,dz$ to $rdr\,d\theta\,dz$, or some other convenient order) and the limits of integration, so as to describe the same region.

   (b) How do you change from rectangular coordinates to spherical coordinates in a triple integral?

   You need to change to integrand (change x to $\rho\sin\phi\cos\theta$; y to $\rho\sin\phi\sin\theta$; z to $\rho\cos\phi$), the differential (change $dx\,dy\,dz$ to $\rho^2\sin\phi d\rho\,d\theta\,d\phi$, or some other convenient order) and the limits of integration, so as to describe the same region.

   (c) In what situations might you change to cylindrical or spherical coordinates?

   You change if that would make the integrand simpler, of if the region of integration is more easily described in cylindrical or spherical coordinates.

2. Use cylindrical coordinates to evaluate $\iiint_E (x^3 + xy^2)\,dV$ where E is the solid in the first octant that lies beneath the paraboloid $z = 1 - x^2 - y^2$.

   As we're in the first octant, θ ranges from 0 to $\pi/2$. r ranges from 0 to 1 (independently of θ). Given a value of r, z ranges from 0 to $1 - r^2$. Hence the integral we want is

$$\begin{aligned}\int_0^{\pi/2}\int_0^1\int_0^{1-r^2}(r^3\cos^3\theta + r^3\cos\theta\sin^2\theta)r\,dz\,dr\,d\theta &= \int_0^{\pi/2}\int_0^1\int_0^{1-r^2} r^4\cos\theta(\cos^2\theta+\sin^2\theta)\,dz\,dr\,d\theta\\ &= \int_0^{\pi/2}\int_0^1 (r^4 - r^6)\cos\theta\,dr\,d\theta\\ &= [\frac{1}{5}r^5 - \frac{1}{7}r^7]_0^1[\sin\theta]_0^{\pi/2}\\ &= \frac{1}{5} - \frac{1}{7}\end{aligned}$$

3. Use a change of variables to evaluate $\iint_R e^{x^2+4y^2}\,dA$ where R is the ellipse $x^2 + 4y^2 \le 1$.

We make the change $x = u, 2y = v$, hoping to be able to use polar later. Solving for x and y gives $x = u, y = v/2$, giving us a Jacobian of $\frac{1}{2}$. $x^2 + 4y^2 \le 1$ is equivalent to $u^2 + v^2 \le 1$ – let this region of the (uv)-plane by R'. Then the integral is equal to

$$\iint_{R'} \frac{e^{u^2+v^2}}{2}\,dA_{(u,v)}.$$

We observe that this is much easier to evaluate in polar coordinates, so we switch to them getting

$$\frac{1}{2}\int_0^{2\pi}\int_0^1 re^{r^2}\,dr\,d\theta = \frac{\pi}{2}\int_0^1 e^t\,dt \qquad t = r^2$$

$$dt = 2r\,dr$$

$$= \frac{\pi}{2}(e-1).$$

Math 53 – Quiz 11 – 8/8/06.

1. If $\mathbf{F} = \langle P, Q\rangle$ is a force field and a particle moves along a smooth curve C given by a vector function $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, $t \in [a, b]$, write an expression for the work done by whatever's causing the forces (eg. a fluid) to help the particle in two ways:

   (a) Using just six symbols (lines drawn under a symbol don't count as a separate symbol);

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

   (b) In terms a first semester calculus student should understand (ie. get rid of all vector notions).

$$\int_a^b \left(P(x(t), y(t))\frac{dx}{dt} + Q(x(t), y(t))\frac{dy}{dt} \right) dt.$$

2. Evaluate $\int_C \langle yz, xz, yz\rangle \cdot d\mathbf{r}$ where C is given by $\mathbf{r}(t) = \langle 1, t, t^2\rangle$, $t \in [0, 1]$.

   $\frac{d}{dt}\mathbf{r}(t) = \langle 0, 1, 2t\rangle$, so the integral we're doing is

$$\begin{aligned} \int_C xy\,dx + xz\,dy + yz\,dz &= \int_0^1 ((t)(0) + t^2(1) + t^3(2t))\,dt \\ &= \int_0^1 (t^2 + 2t^4)\,dt \\ &= \frac{1}{3} + \frac{2}{5} \end{aligned}$$

3. A particle is dropped into a 2D fluid with velocity field $\langle e^y, e^{-x}\rangle$ at the origin. Along what curve does it flow?

The motion of the particle will follow the differential equations $\dot{x} = e^y$, $\dot{y} = e^{-x}$. Dividing the second equation by the first and separating variables, we get

$$\begin{aligned} \frac{dy}{dx} &= e^{-x}e^{-y} \\ \int e^y\,dy &= \int e^{-x}\,dx \\ e^y &= -e^{-x} + C \end{aligned}$$

$(0, 0)$ lies on the curve, so we must have $C = 2$. Hence the equation of the curve is $y = \ln(2 - e^{-x})$ (although, actually, only the right hand half of this is traversed).

Math 53 – Quiz 12 – 8/11/06.

1. Give a clear and precise statement of Green's Theorem. [Remember, the conditions are as important as the conclusion.]

   Green's Theorem states the following: Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivates on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

2. Find a function f such that $\nabla f = \mathbf{F} = \langle e^y - y\sin x, xe^y + \cos x + 2y\rangle$ and use it to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the upper semi-circle from $(0,-1)$ to $(2\pi,1)$.

   We know that

$$f_x = e^y - y\sin x$$
$$f_y = xe^y + \cos x + 2y.$$

   Integrating these gives,

$$f = xe^y + y\cos x + g(y)$$
$$f = xe^y + y\cos x + y^2 + h(x)$$

   Comparing these, we see a choice that works for f is $f(x,y) = xe^y + y\cos x + y^2$.

   Hence,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= f(2\pi,1) - f(0,-1)\\ &= 2\pi e + 2\end{aligned}$$

3. Draw a (2D) picture of the velocity field of some water near a whirl pool. Is this field conservative? Explain your answer.

   The picture should give flow lines which are spirals in towards some point, say the origin. These may be clockwise or counterclockwise, let's suppose they're counter. The field is not conservative. To see this, pick two points which are "opposite one another", say $A = (0,1)$ and $B = (0,-1)$. Consider two paths from A to B: C_1 is the left semicircle connecting them, C_2 the right one. Then, by inspection, $\int_{C_1} \mathbf{F}\cdot d\mathbf{R}$ is positive and $\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ is negative. However, these two paths have the same endpoints, so, if $\mathbf{F}$ were conservative, they'd be equal by the Fundamental Theorem for Line Integrals. Hence, $\mathbf{F}$ is not conservative.

Math 53 – Quiz 13 – 8/15/06.

1. (a) What is an oriented surface? Give an example of a non-orientable surface.

   An oriented surface is a surface with one side designated the 'front'. The Möbius band is a non-orientable surface.

   (b) How do you evaluate a surface integral of a vector field **F** over a surface given by a vector function $\mathbf{r}(u,v)$, $(u,v) \in D$. If **F** is the velocity field of a fluid, what does the surface integral represent?

   The surface integral can be found by evaluating the following double integral:

$$\iint_D \mathbf{F}(x(u,v), y(u,v), z(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.$$

   If **F** is the velocity field of a fluid and the surface is a permeable membrane, then the integral is the net rate at which fluid is flowing from the front to the back of it.

2. Parametrise the surface $z = xy$ and hence find the area of the part which lies inside the cylinder $x^2 + y^2 = 1$.

   The easiest way to parametrize this surface is as $\mathbf{r}(u,v) = \langle u, v, uv \rangle$. The part of the surface we're interested in is $(u,v) \in D$ where D is the unit circle. Calculating, we have:

$$\begin{aligned} \mathbf{r}_u &= \langle 1, 0, v \rangle \\ \mathbf{r}_v &= \langle 0, 1, u \rangle \\ \therefore \mathbf{r}_u \times \mathbf{r}_v &= \langle -v, -u, -1 \rangle \\ \therefore |\mathbf{r}_u \times \mathbf{r}_v| &= \sqrt{1 + u^2 + v^2} \end{aligned}$$

   So, the integral we want is

$$\iint_D \sqrt{1 + u^2 + v^2}\, dA$$

   which is easiest evaluated in polar coordinates:

$$\begin{aligned} \int_0^{2\pi} \int_0^1 r\sqrt{1 + r^2}\, dr\, d\theta = \pi \int_1^2 \sqrt{t}\, dtt & \qquad = 1 + r^2 \\ & \qquad dt = 2r\, dr \\ = \frac{2\pi}{3} t^{3/2}]_1^2 & \\ = \frac{2\pi}{3}(2^{3/2} - 1). & \end{aligned}$$

3. Write $\mathbf{F} = \langle x^2 + yz, \cos(x+z) + \sin(y), z + xe^{-y} \rangle$ as the some of an irrotational field and an incompressible one.

Recall from homework that any field of the form $\langle f(x), g(y), h(z) \rangle$ is irrotational and from class that any field of the form $\langle f(y,z), g(x,z), h(x,y) \rangle$ is incompressible. Luckily, **F** can be broken down like that:

$$\mathbf{F} = \langle x^2, \sin(y), z \rangle + \langle yz, \cos(x+z), xe^{-y} \rangle.$$